Breaking the Quadratic Barrier for Matroid Intersection

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Summary

Result:

First subquadratic independence-query matroid intersection algorithm.

- Previous best: $\tilde{O}(n^2)$ queries.
- **Ours:** $\tilde{O}(n^{9/5})$ randomized and $\tilde{O}(n^{11/6})$ deterministic.

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Technique:

Previous work + a **new** simple subquadratic reachability algorithm.

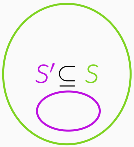
- Previous best: $O(n^2)$ queries.
- **Ours:** $\tilde{O}(n^{3/2})$ randomized and $\tilde{O}(n^{5/3})$ deterministic.

What is a Matroid?

- Set of elements V. $n = |V|$.
- Notion of independence $\mathcal{I} \subseteq 2^V$.

Downward Closure

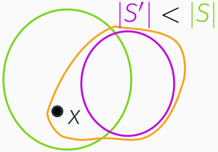
$S \in \mathcal{I}$



$\Rightarrow S' \in \mathcal{I}$

Exchange Property

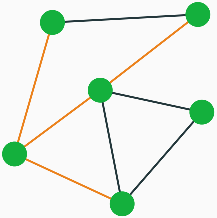
$S \in \mathcal{I}$ $S' \in \mathcal{I}$



$\Rightarrow \exists x \in S \setminus S'$ such that $S' \cup \{x\} \in \mathcal{I}$

Matroids — Examples

Graphic Matroid



$V = \text{edges}$

$\mathcal{I} = \text{forests}$

Linear Matroid

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & 0 & 1 & 2 & 0 \\ 2 & 0 & 2 & 4 & 0 \\ 1 & 1 & 3 & 2 & 1 \\ 0 & 0 & 1 & 0 & 5 \end{bmatrix}$$

$V = \text{row vectors}$

$\mathcal{I} = \text{linearly independent}$

Matroid Intersection

Given: two matroid $\mathcal{M}_1 = (V, \mathcal{I}_1)$ and $\mathcal{M}_2 = (V, \mathcal{I}_2)$

Goal: find a *common independent set* $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ of maximum size.

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Independence oracle queries: *Is* $X \in \mathcal{I}_1$*?* *Is* $X \in \mathcal{I}_2$*?*

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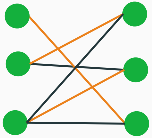
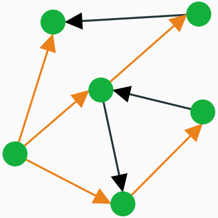
Intersection of **three** matroids is NP-hard.

Matroid Intersection — Examples

Models many combinatorial optimization problems

- Bipartite matching
  - $\mathcal{M}_1 =$ "≤ 1 edge per vertex on the left"
  - $\mathcal{M}_2 =$ "≤ 1 edge per vertex on the right"
- Arborescence (directed spanning tree)
- Colorful spanning trees
- Tree packing
- Graph orientation problems
- ...





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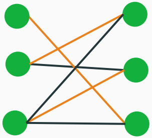

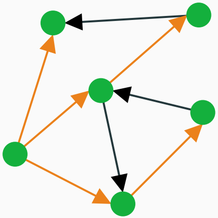

Also connections to Submodular Function Minimization

Previous work

1960s-70s Edmonds, Lawler and Aigner-Downling: $O(n^3)$ queries

- Finding augmenting paths in the *exchange graph*.

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2019 Chakrabarty-Lee-Sidford-Singla-Wong and Nguyễn: $\tilde{O}(n^2)$

- Efficient implementations of Cunningham's algorithm.

Major open problem:

Can we break this Quadratic Barrier?

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YES, with a more powerful rank-oracle.

- Algorithm using $\tilde{O}(n^{1.5})$ rank-queries. [CLSSW 2019]

Queries

Independence: Is $X \in \mathcal{I}$?

Rank: What is $\max_{Y \subseteq X,\ Y \in \mathcal{I}} |Y|$?

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YES, for a $(1-\varepsilon)$-approximate solution.

- Algorithm using $\tilde{O}(n^{1.5}/\varepsilon^{1.5})$ independence-queries. [CLSSW 2019]

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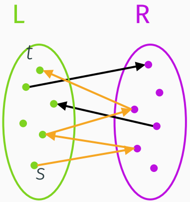
- Randomized: $\tilde{O}(n^{9/5})$ independence-queries.
- Deterministic: $\tilde{O}(n^{11/6})$ independence-queries.

Proof outline

Reachability Problem

Given: Directed bipartite graph G with bipartition (L, R);
Two vertices $s, t \in \mathsf{L}$.

Goal: Find an (s, t)-path, or determine none exist.



Reachability Problem

Given: Directed bipartite graph G with bipartition (L, R);
Two vertices $s, t \in$ L.

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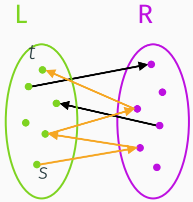
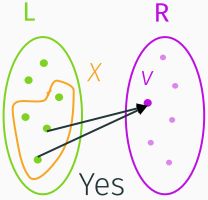
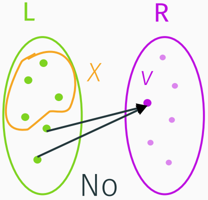
Queries: Specify $v \in$ R and $X \subseteq$ L and ask:

- Does v have an in-neighbor from X?
- Does v have an out-neighbor to X?







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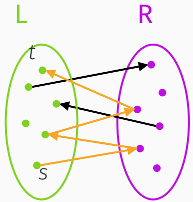
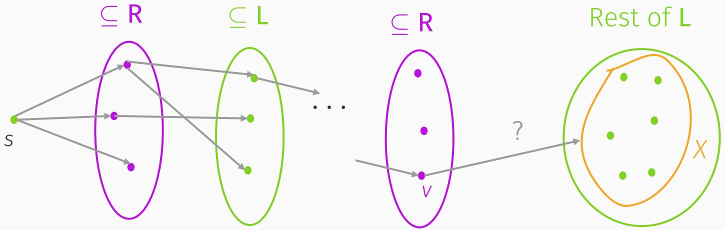


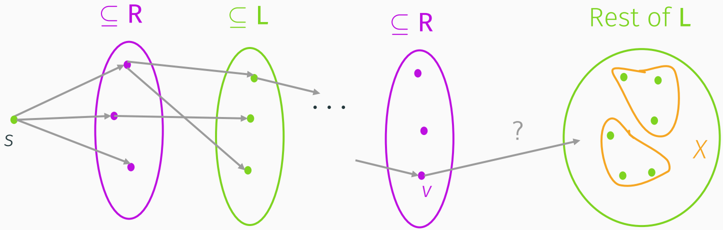
Theorem: Subquadratic **Reachability Problem**
$\implies$ Subquadratic **Matroid Intersection**.

Idea: Many short paths — CLSSW approximation algorithm
Few long paths — Reachability problem.



Allowed Queries: Does $v \in$ **R** have an {out/in}-neighbor from $X \in$ **L**?

First try: Breadth First Search — from R to L



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Binary-search: $O(\log n)$

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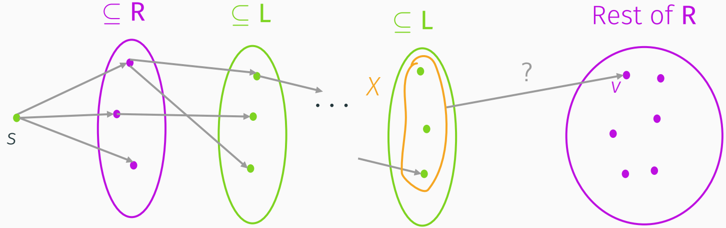

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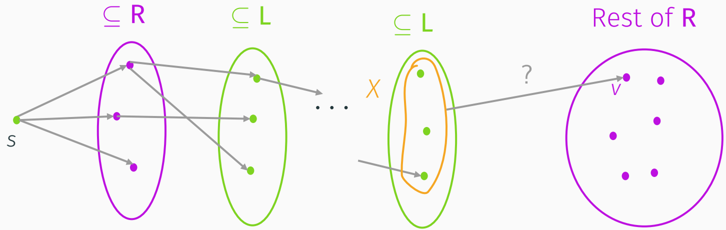

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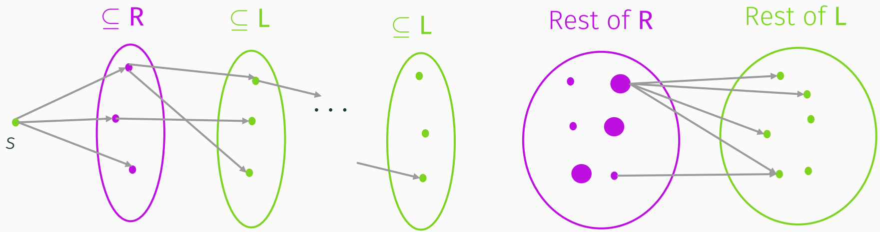
Need $\Omega(n)$ queries
Total: $\Theta(n^2)$ queries

Can we do better?

Heavy and Light Vertices



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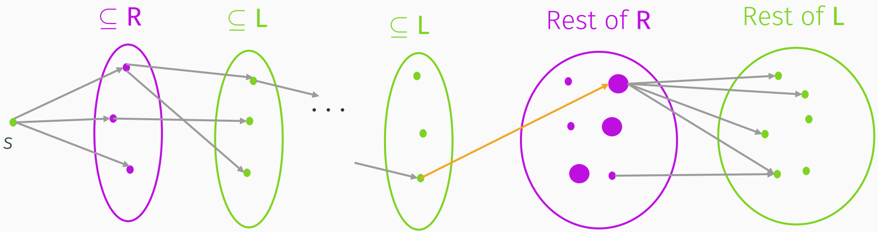

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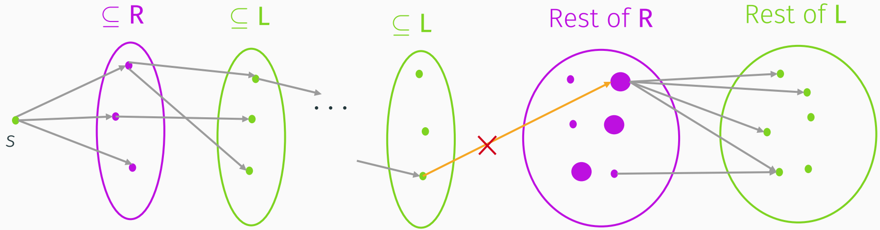

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But what if next layer consists of only light vertices?

Our main insight: We can still efficiently find a *heavy* vertex!

Reverse BFS

- Light vertices have only $O(\sqrt{n})$ outgoing edges. **Find all of them!**

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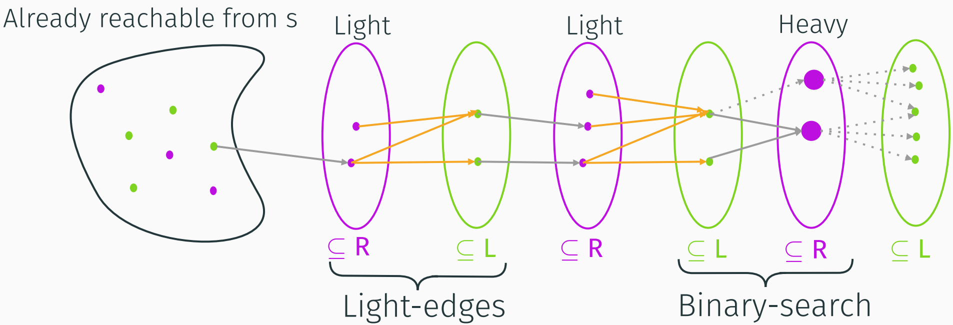

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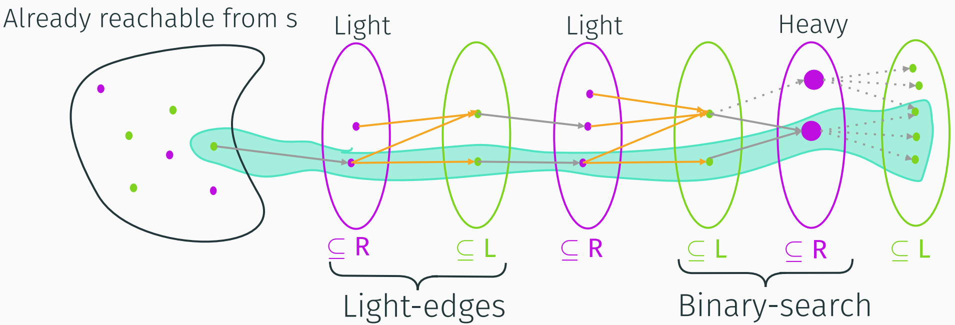

Reachability Problem — Algorithm

Run in $O(\sqrt{n})$ phases:

- Categorize heavy / light

- Find all outgoing edges of newly light vertices
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Total Query Complexity: $\tilde{O}(n\sqrt{n})$ randomized or $\tilde{O}(n^{5/3})$ deterministic.

Summary

- **Reachability Problem**: subquadratic number of queries.
  - Previous best: $O(n^2)$ queries.
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Previous work + subquadratic Reachability Problem $\implies$

- **Matroid Intersection**: subquadratic number of *independence-queries*.
  - Previous best: $\tilde{O}(n^2)$ queries.
  - **Ours:** $\tilde{O}(n^{9/5})$ randomized and $\tilde{O}(n^{11/6})$ deterministic.

Open Problems

- Gap between lower and upper bounds for matroid intersection.
    - No $\Omega(n^{1+\epsilon})$ lower-bound is known for $\epsilon > 0$.
- Tight bounds for the reachability problem.
  We conjecture that our $\tilde{O}(n\sqrt{n})$ bound is tight.
- Can one also solve **weighted** matroid intersection with subquadratic number of queries?
- Investigating the gap between *independence* and *rank* oracle models.
    - Reachability Problem: $O(n\sqrt{n})$ vs $O(n)$.
    - Approximate Matroid Intersection: $O(n\sqrt{n}/\text{poly}(\varepsilon))$ vs $O(n/\varepsilon)$.

Thanks!

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