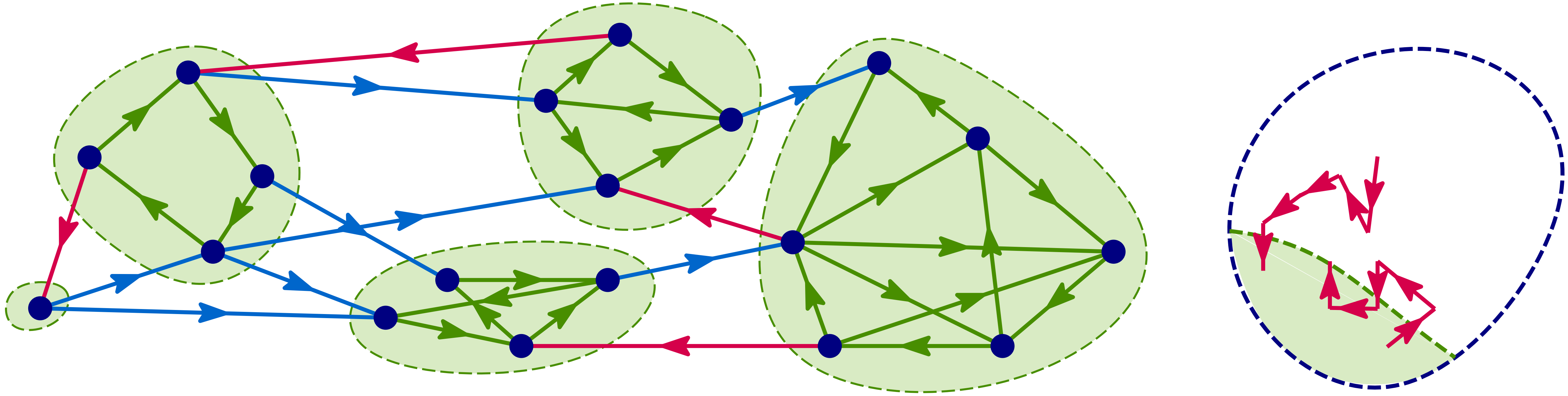
Combinatorial Maxflow in $n^{2+o(1)}$ Time

Aaron Bernstein
Joakim Blikstad[†]
Thatchaphol Saranurak
Ta-Wei Tu

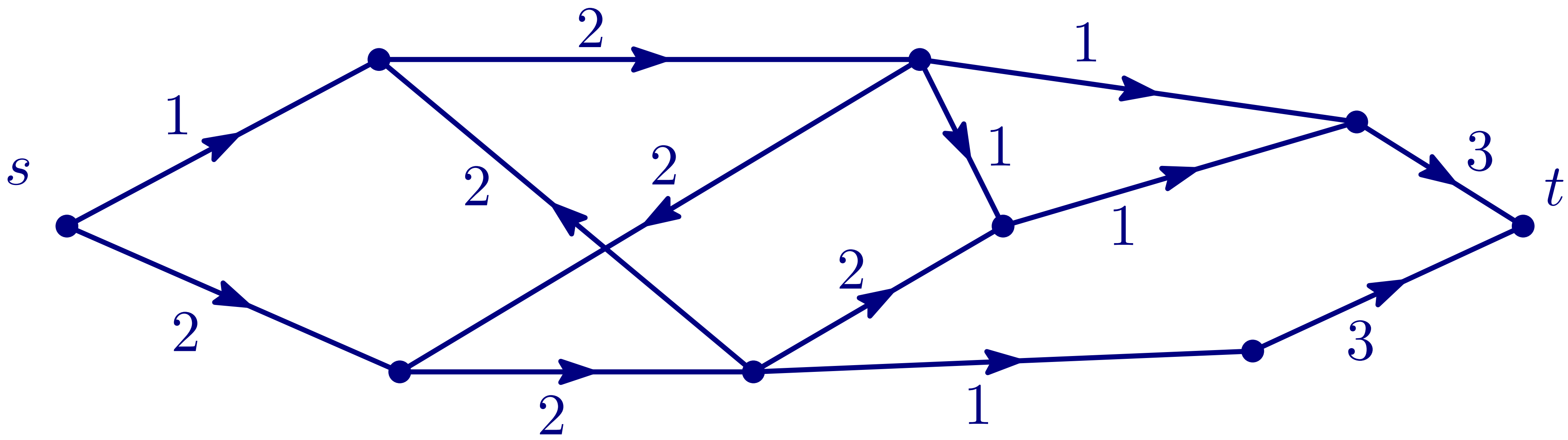


Algorithms & Complexity @ Warwick; Sep, 2024

[†]KTH Royal Institute of Technology

Maximum Flow

Given: Directed graph $G = (V, E)$, edge capacities $c : E \to \mathbb{Z}_{\geq 1}$, source s, and sink t.
Goal: Compute s, t-flow f of largest size.



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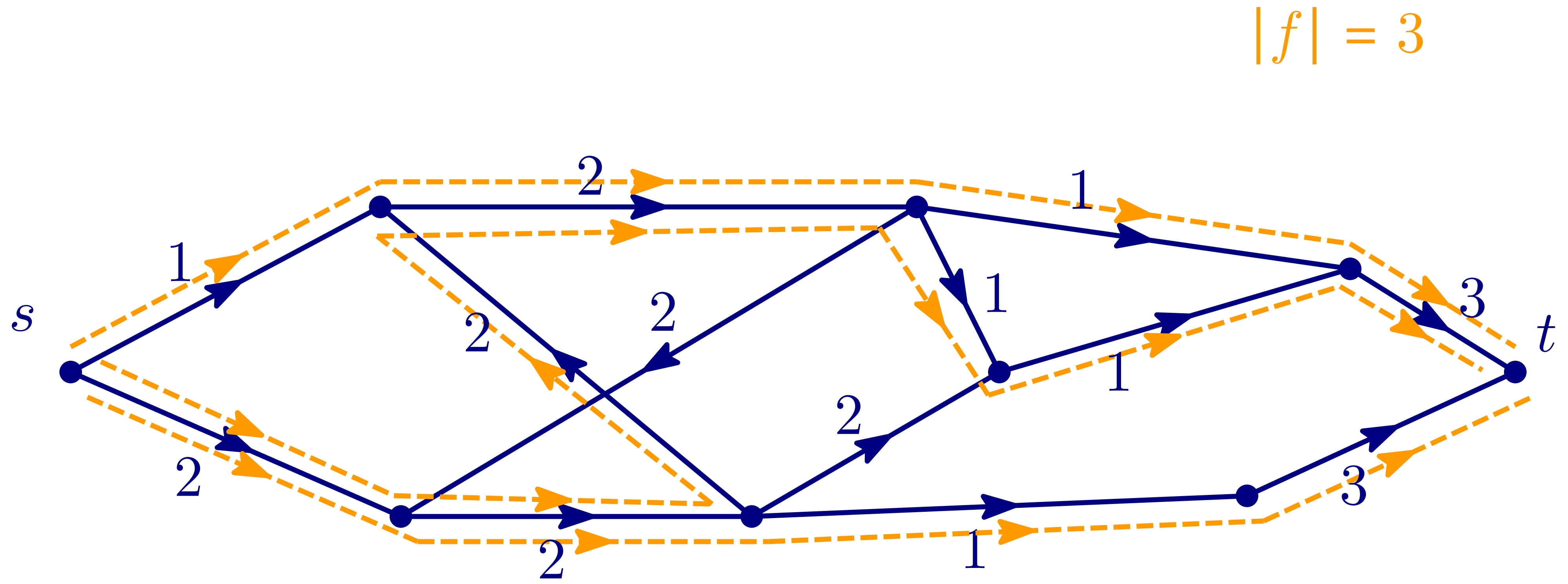

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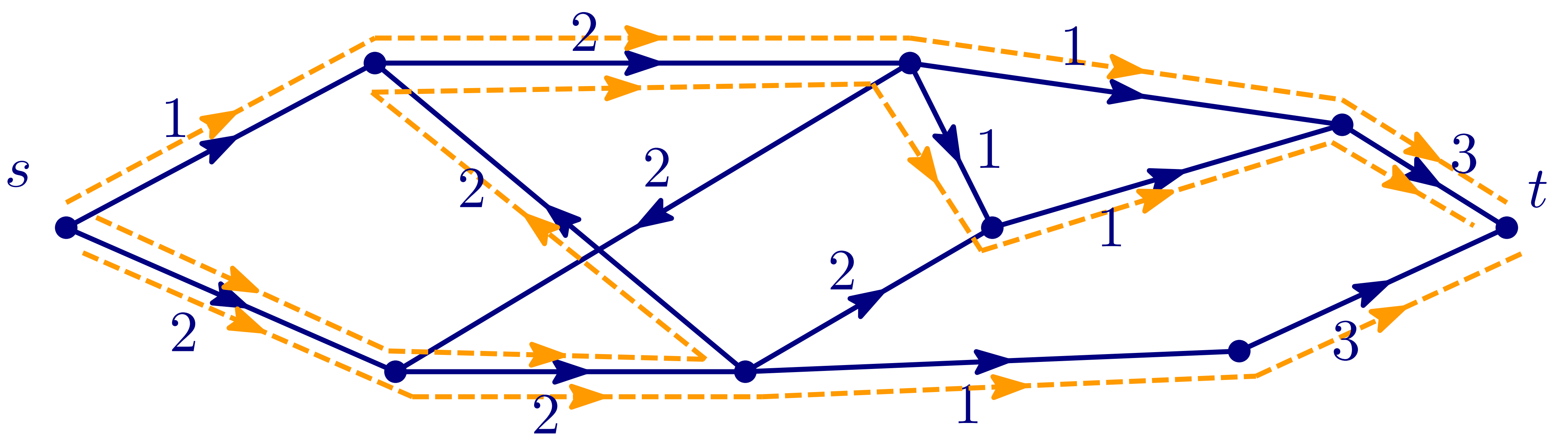
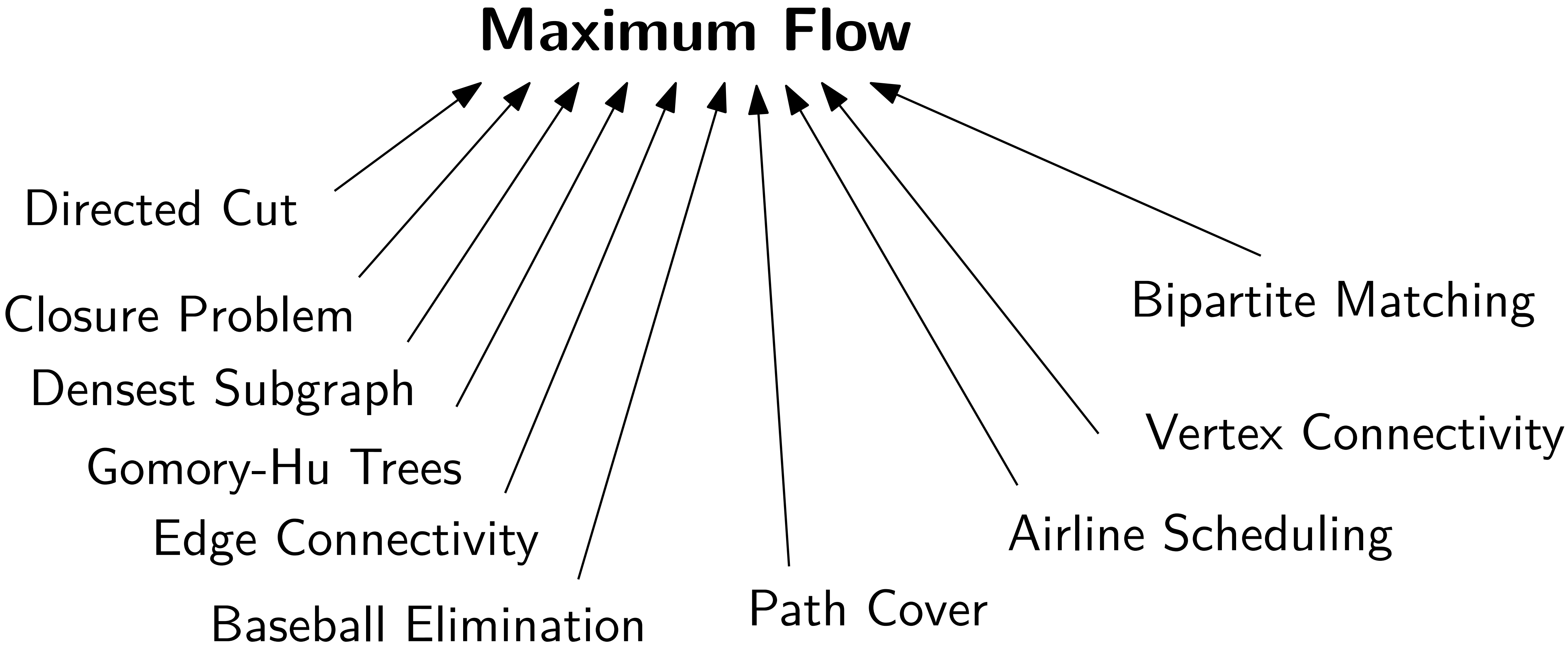
Flow satisfies:

(1) Capacity constraints $f(e) \leq c(e)$
(2) Conservation of flow "incoming $=$ outgoing"

$|f| = 3$



Reductions


Maximum Flow
Directed Cut
Closure Problem
Densest Subgraph
Gomory-Hu Trees
Edge Connectivity
Baseball Elimination
Path Cover
Airline Scheduling
Vertex Connectivity
Bipartite Matching

Maximum Flow — A Brief History

1955	Ford-Fulkerson	$O(E \cdot \|\text{answer}\|)$
1970	Edmonds-Karp	$O(VE^2)$
1970	Dinic "Blocking Flow"	$O(V^2E)$
1978	Malhotra-Kumar-Maheshwari	$O(V^3)$
1983	Dinics Dynamic Trees	$\tilde{O}(VE)$
1986	Goldberg-Tarjan "Push-Relabel"	$\tilde{O}(V^3)$
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$\tilde{O}$: hides polylog

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Combinatorial, Augmenting Paths

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Continiuous Optimization, Interior Point Methods

"Precision issues", "I don't understand them"

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2024	Bernstein-**B.**-Saranurak-Tu	$\hat{O}(V^2)$ "linear time in dense graphs"

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New Era? — Comeback of Combinatorial Algorithms

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Main Result: Maximum flow in on n-vertex graphs in $n^{2+o(1)}$ time.

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Independent Work: $n^{2+o(1)}$ combinatorial bipartite matching [Chuzhoy-Khanna'24]

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Techniques:

Augmenting Paths (new version of Push-Relabel)

Directed Expander Hierarchy

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My Hope: (in a few years)

"Simple", "Combinatorial" $\tilde{O}(E)$ Maximum Flow?

Non-bipartite Maximum Matching in $\tilde{O}(V^2)$ or $\tilde{O}(E)$ time?

Independent Work:
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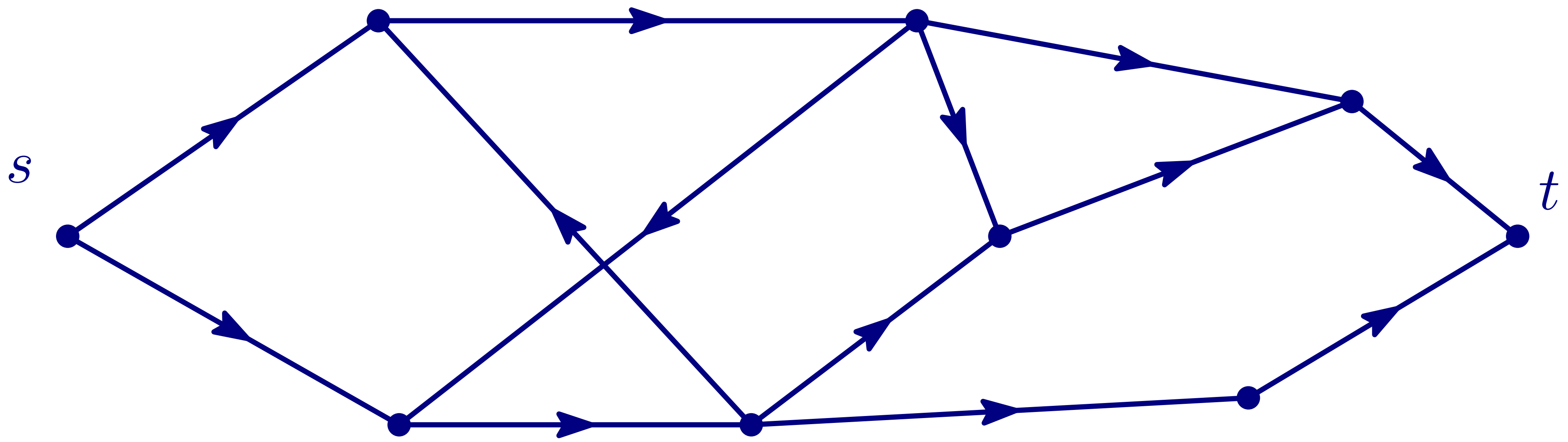
Augmenting Paths

[Ford-Fulkerson 1955]
[Jacobi 1836]



Augmenting Paths

[Ford-Fulkerson 1955]
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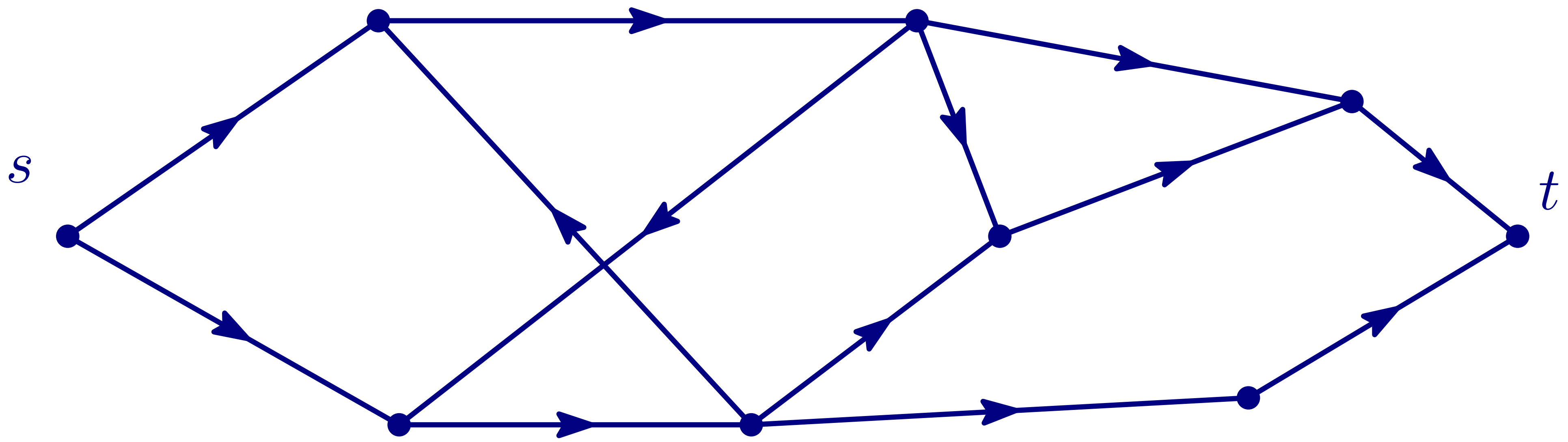
Remainder of this talk: unit-capacities $c(e) = 1$



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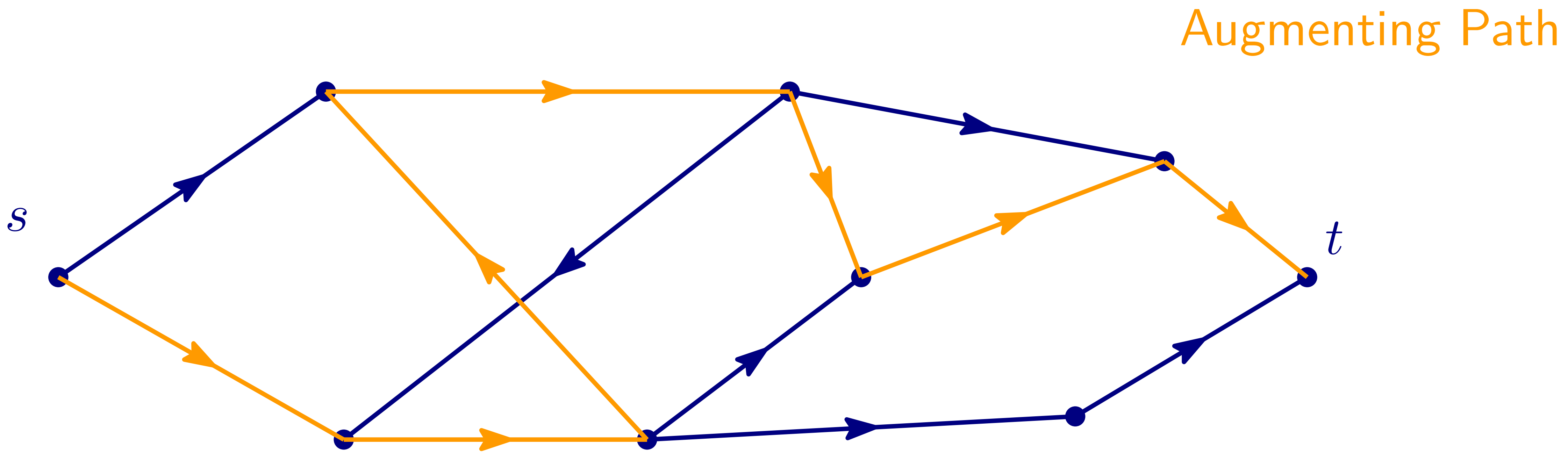

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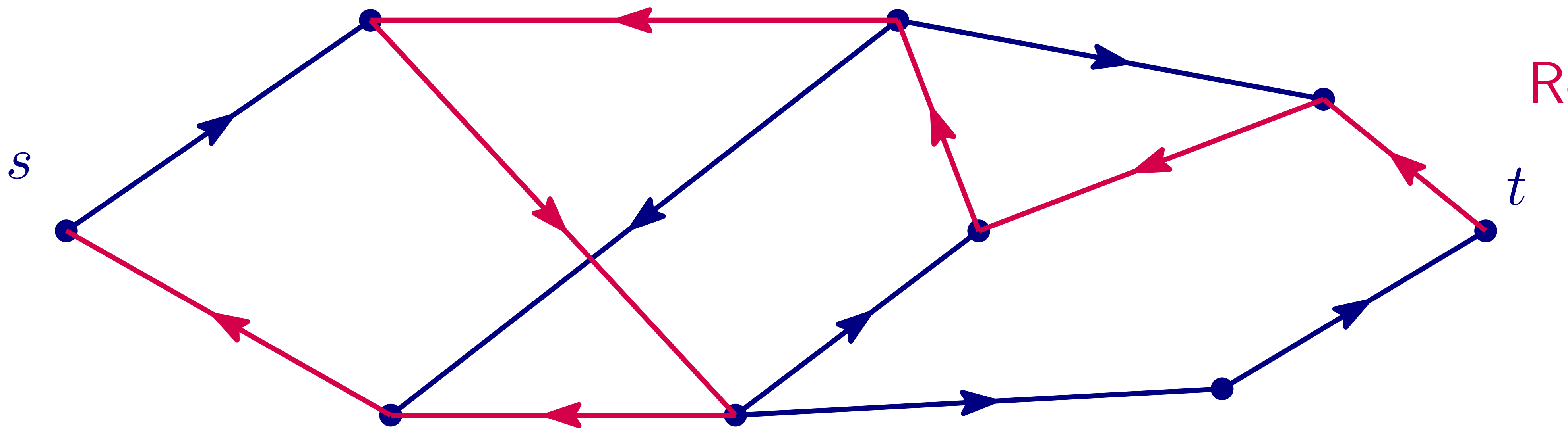

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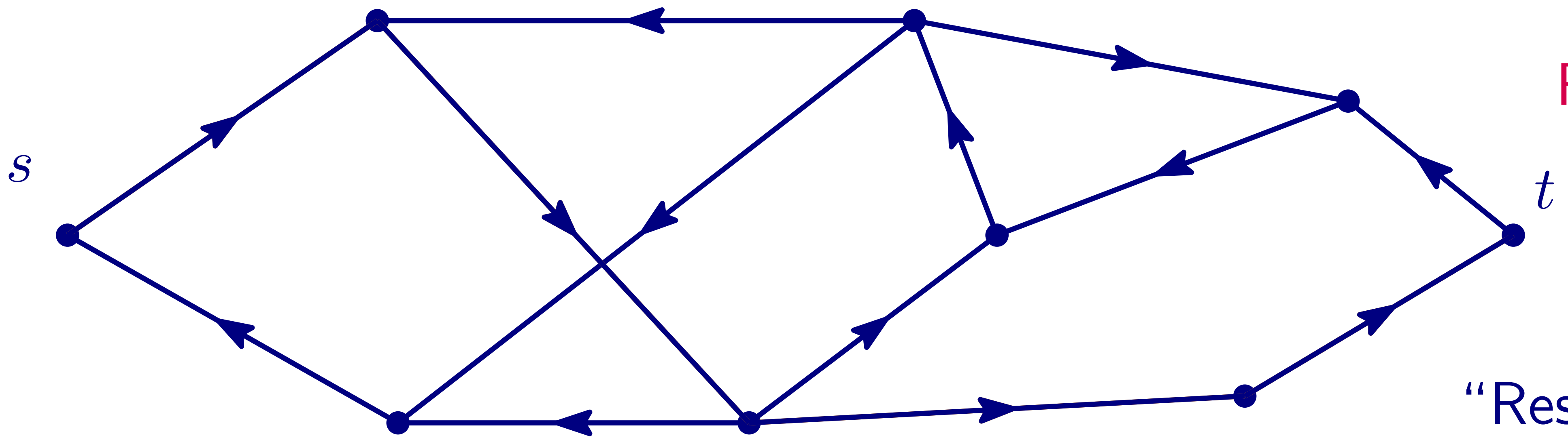
Reverse it



"Residual Graph"

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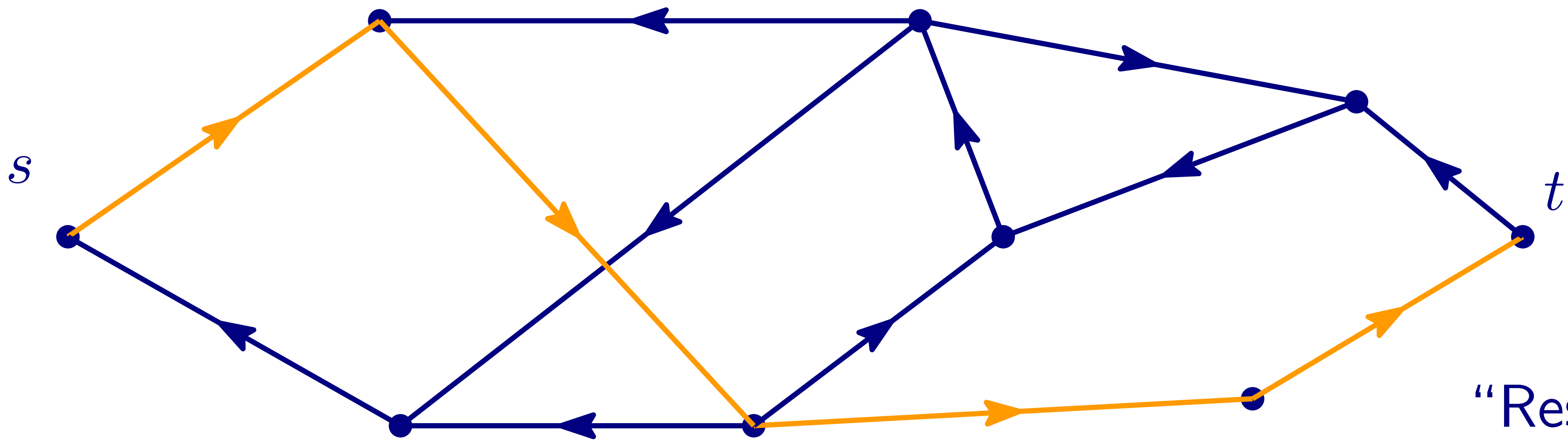

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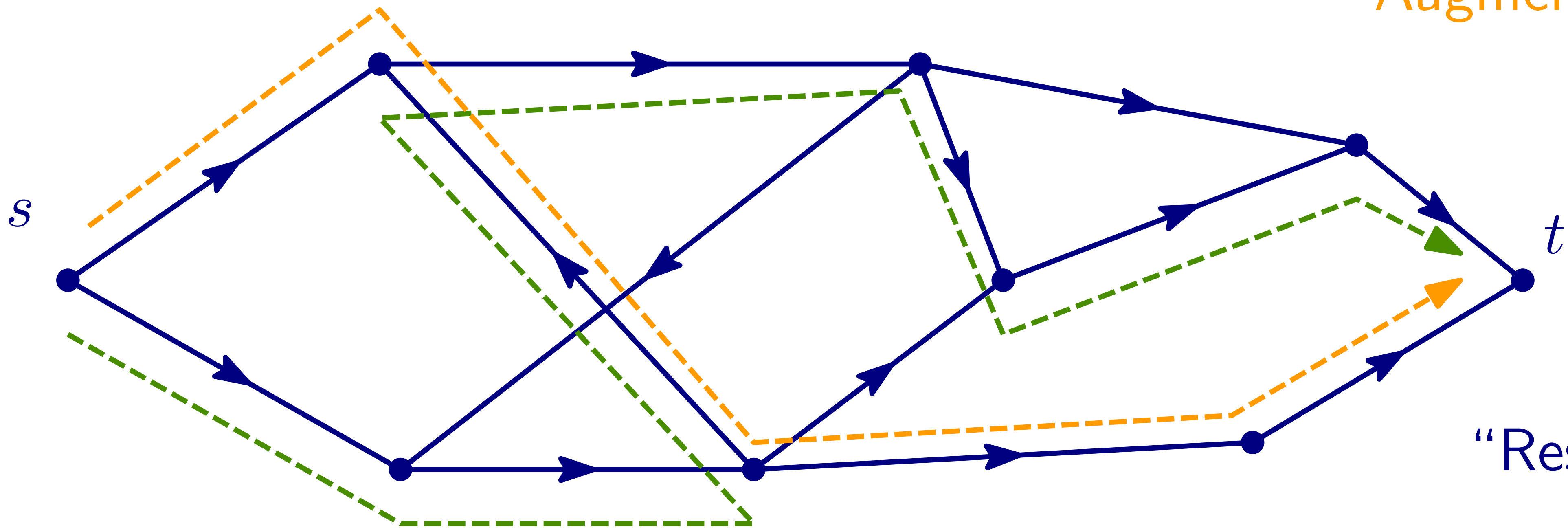

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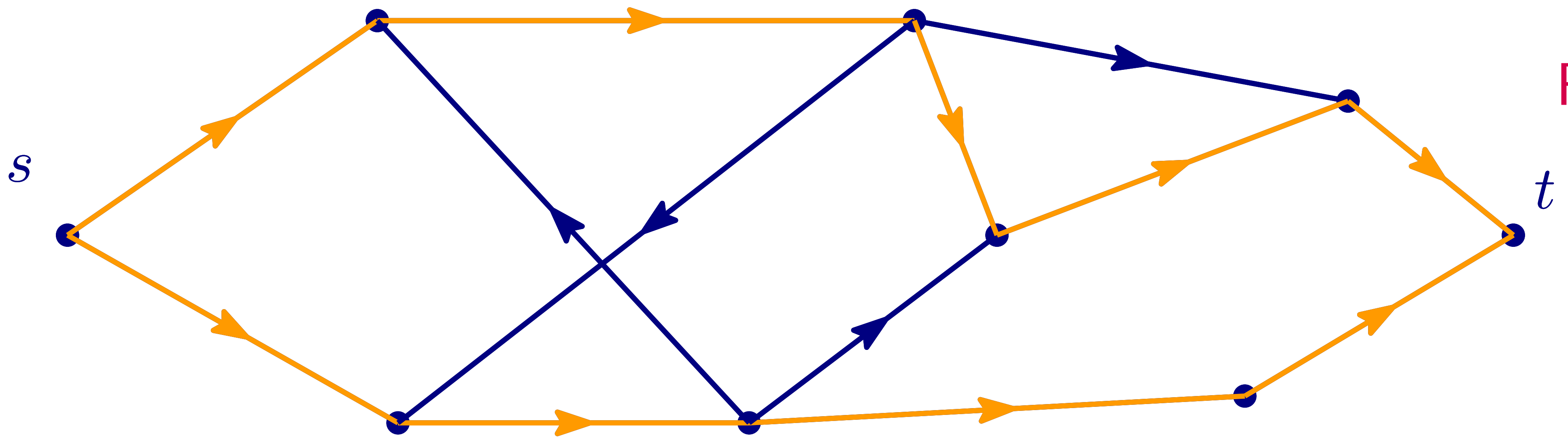


Maximum Flow f, $|f| = 2$

Approximate Flow $\Longrightarrow$ Exact Flow

Approximate Flow $\implies$ Exact Flow

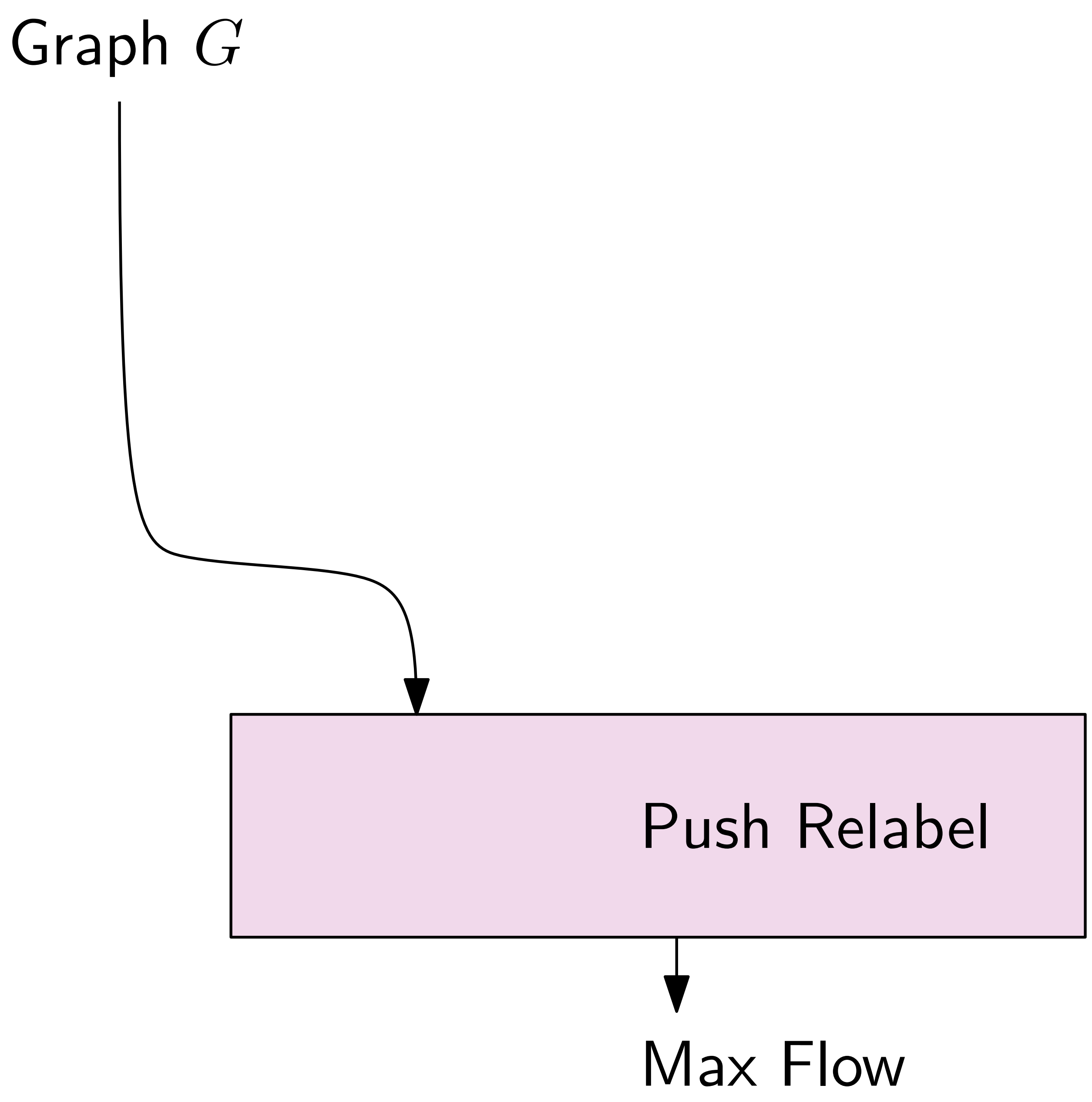
Proof. Recurse on residual graph.

Goal in rest of talk: constant- or $\frac{1}{n^{o(1)}}$-approx flow.

(does not work in undirected graphs)

Outline

1. Recap: Push-Relabel



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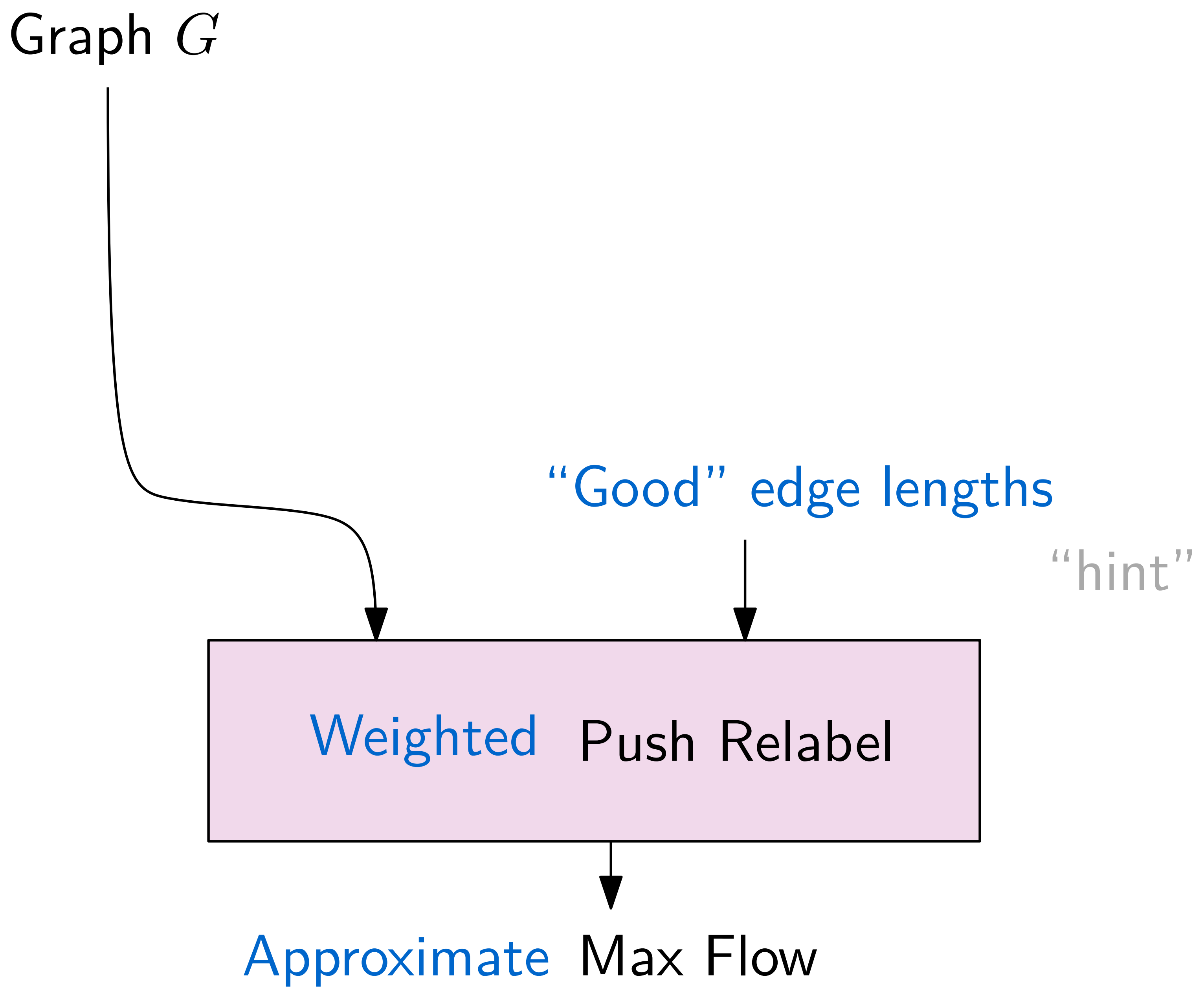

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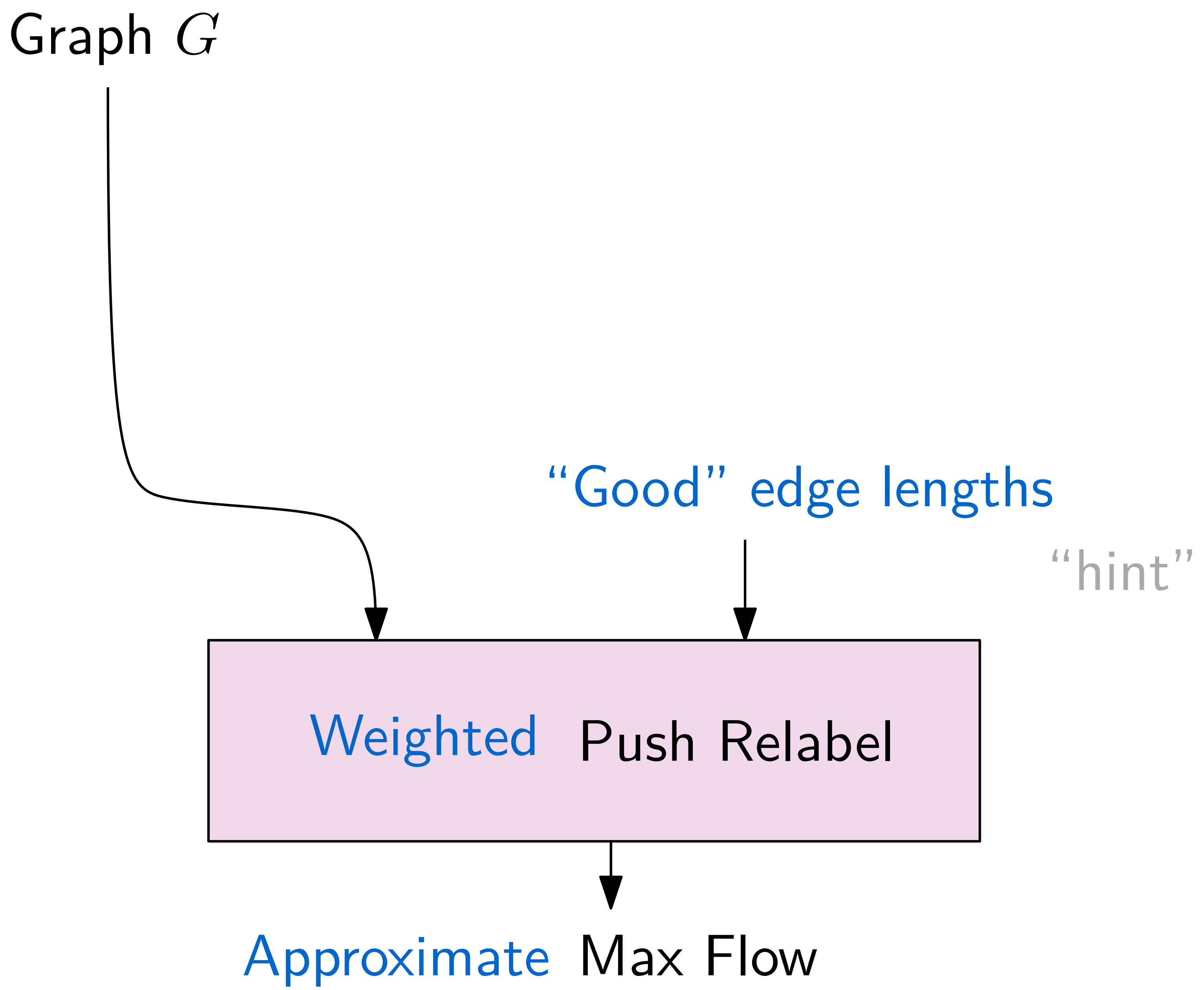

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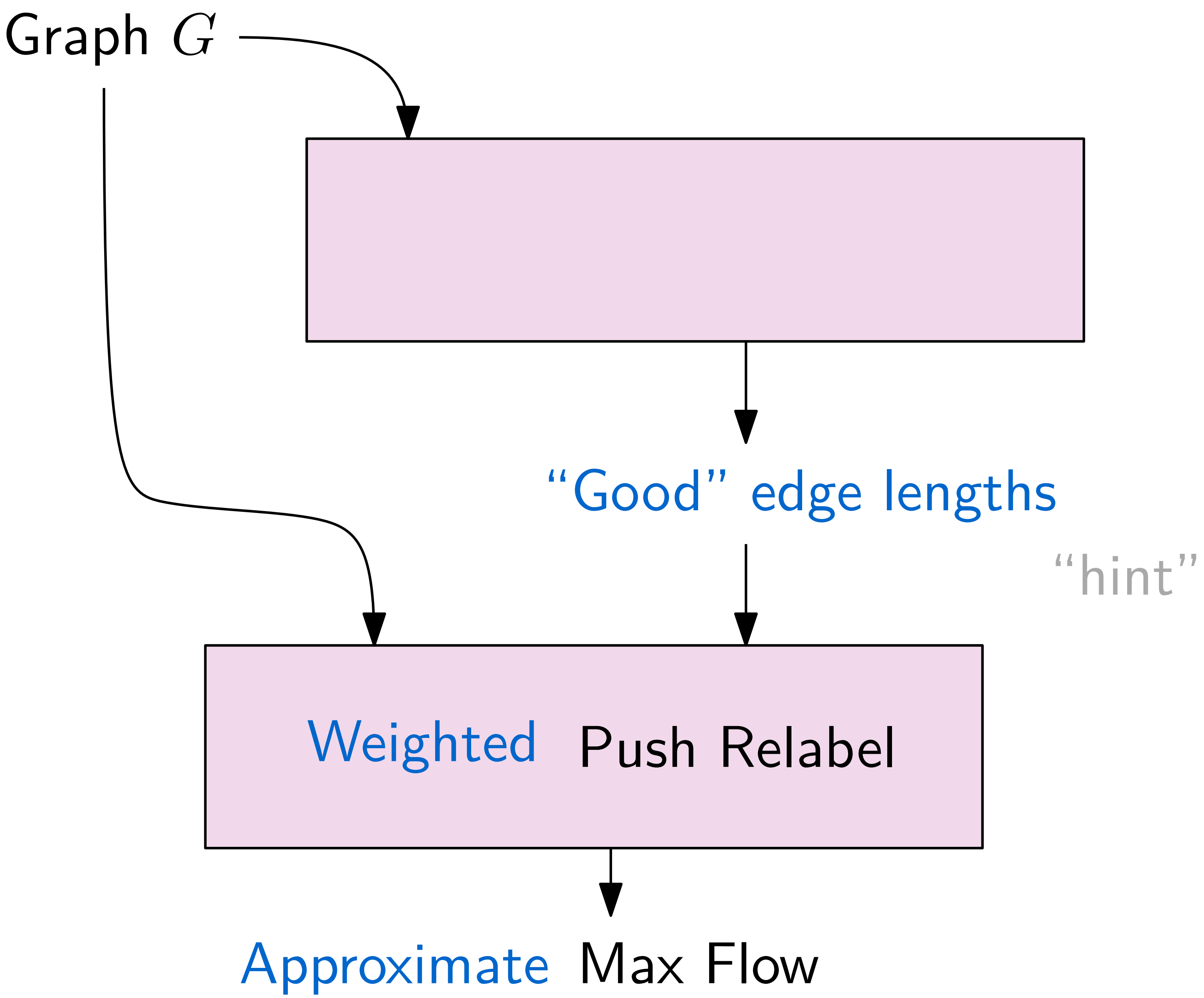

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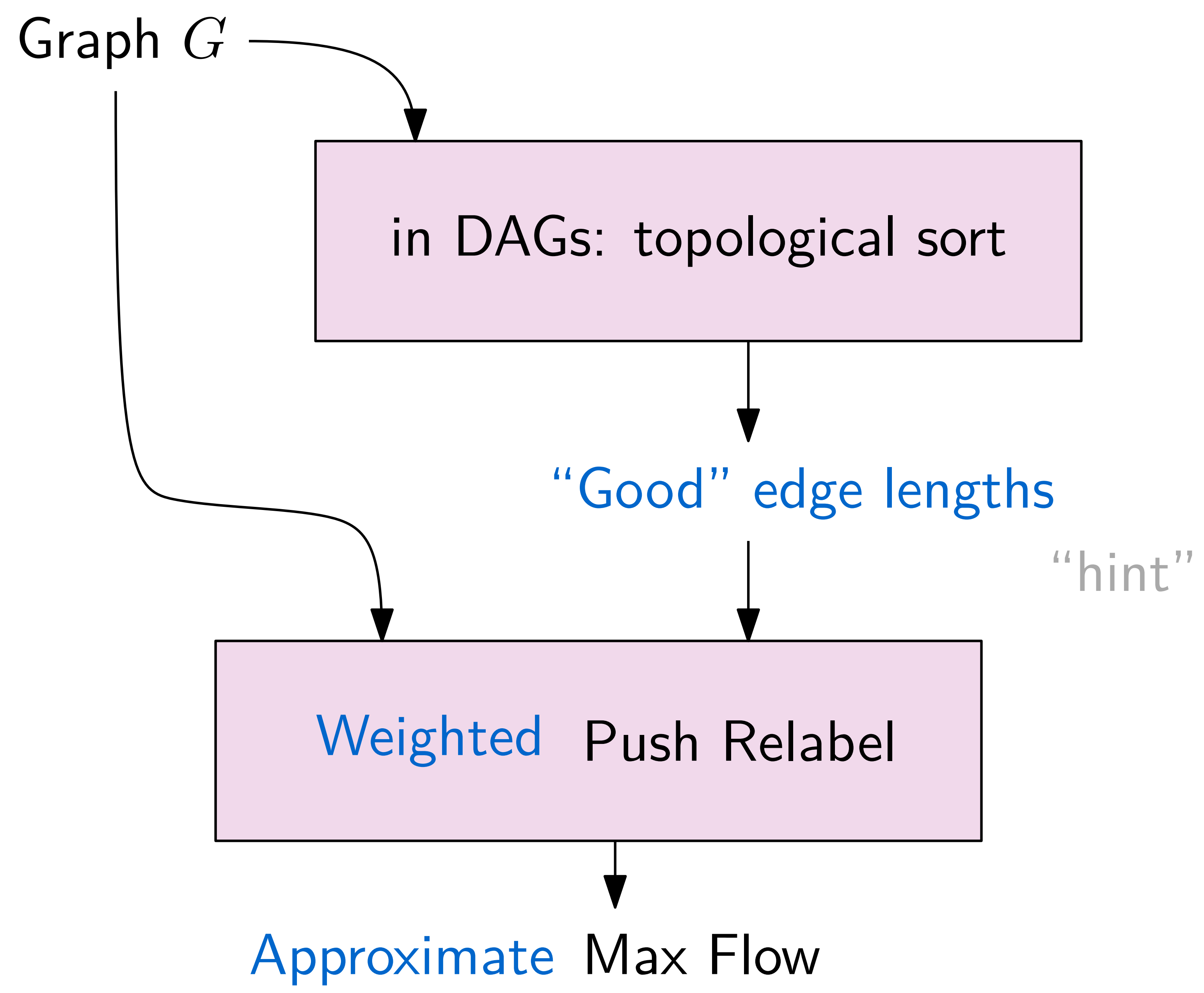
4. Edge Lenghts in DAGs



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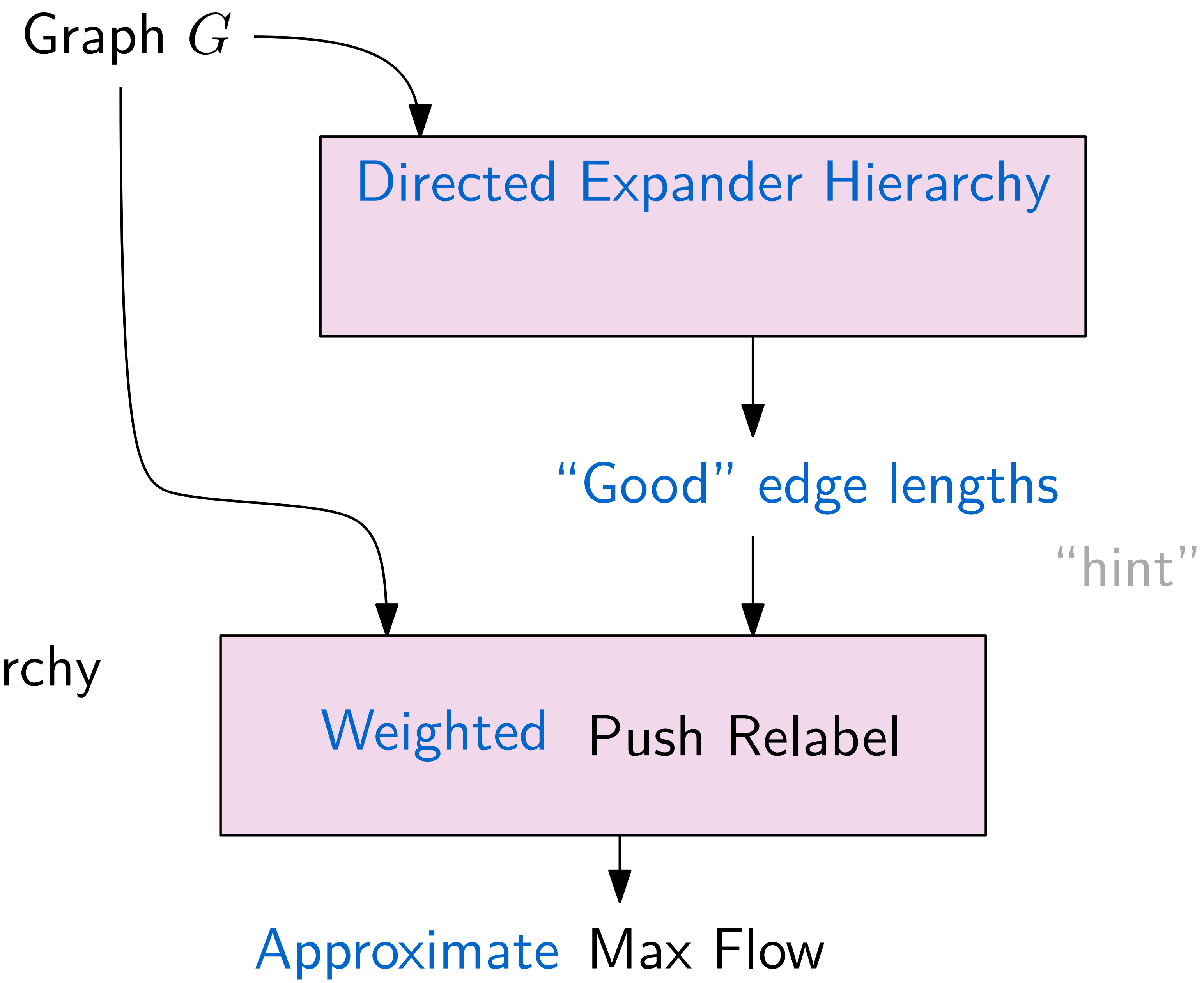

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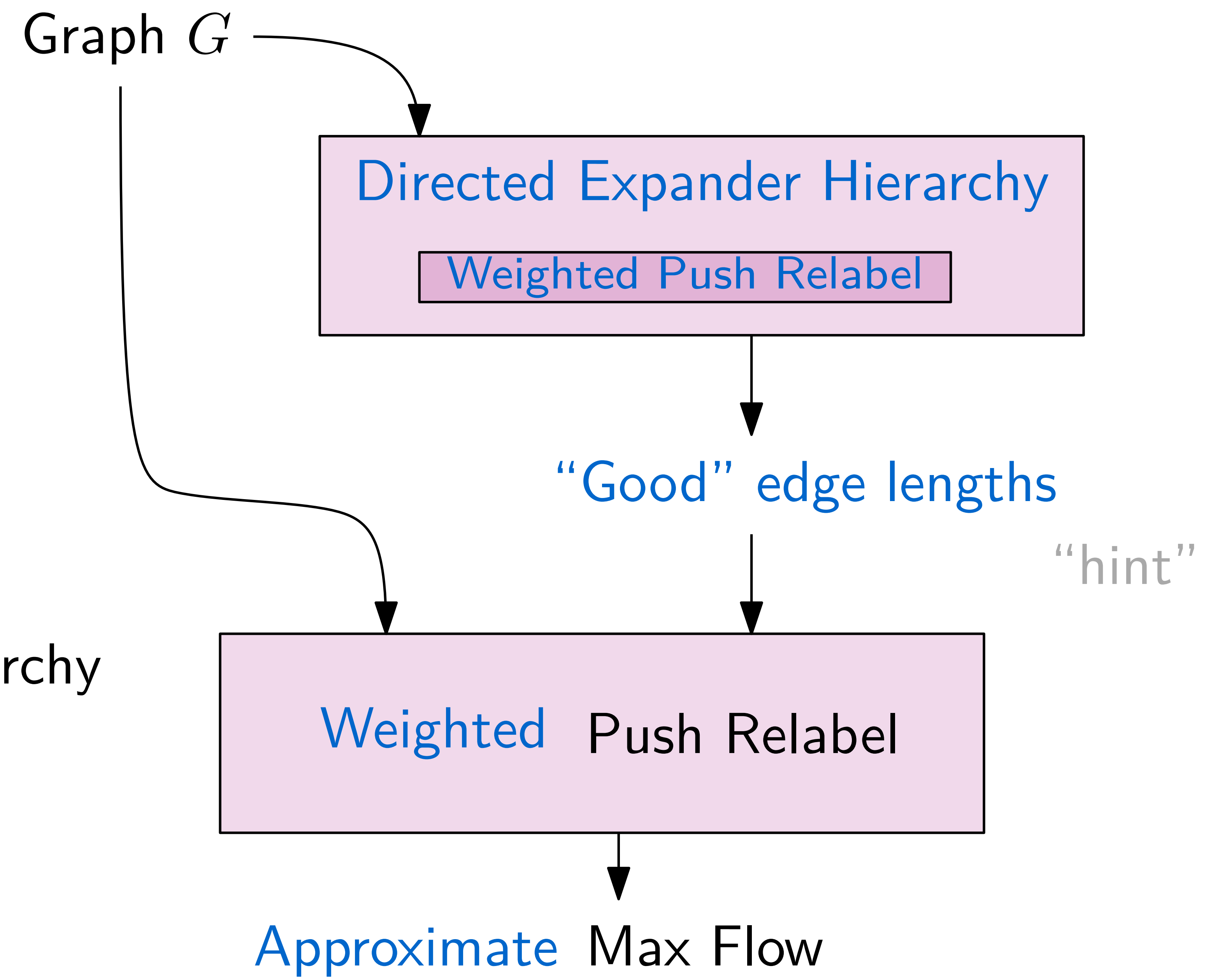
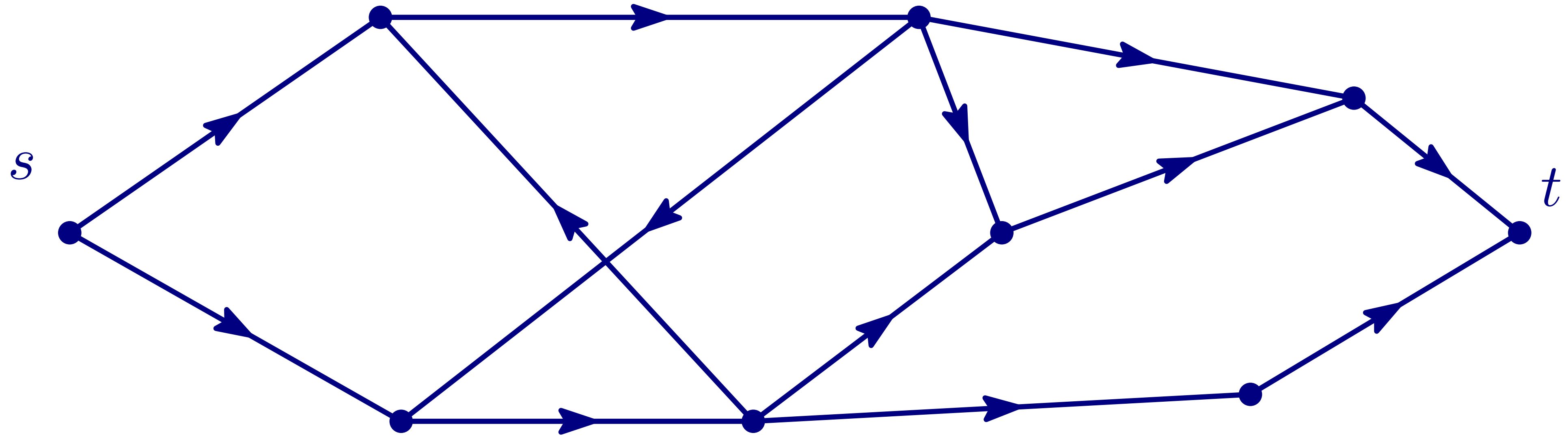


Push-Relabel / Augment-Relabel [Goldberg-Tarjan'88]



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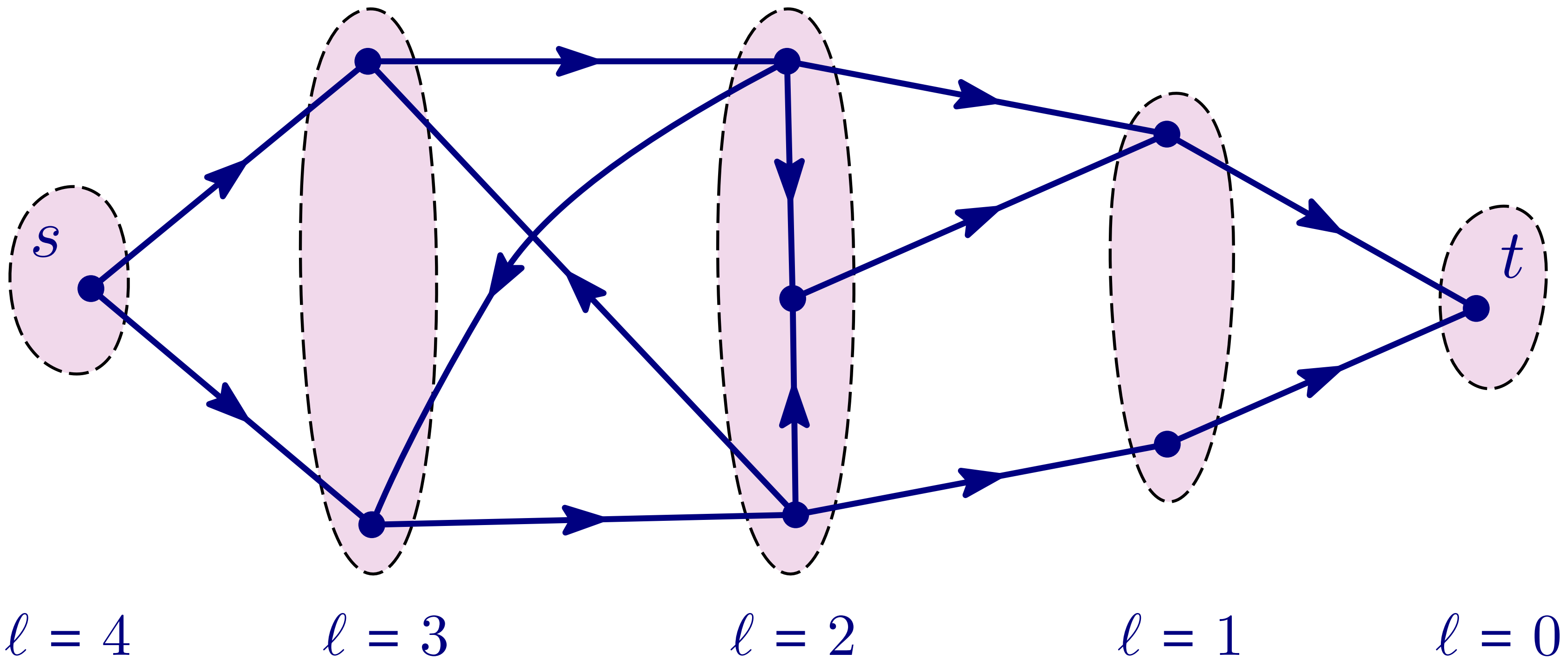
$\ell(v) = \text{dist}(v, t)$



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edge $e = (u, v)$ *admissible* iff $\ell(u) = \ell(v) + 1$

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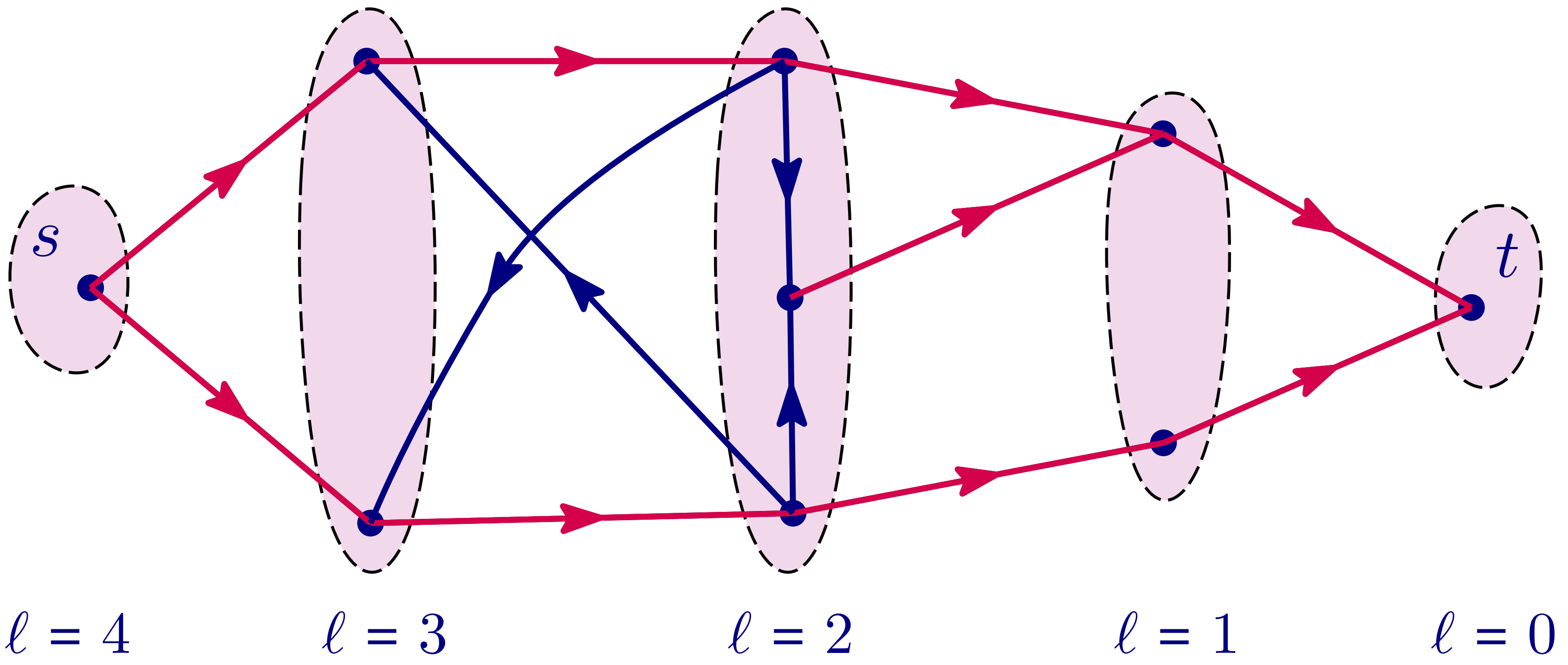

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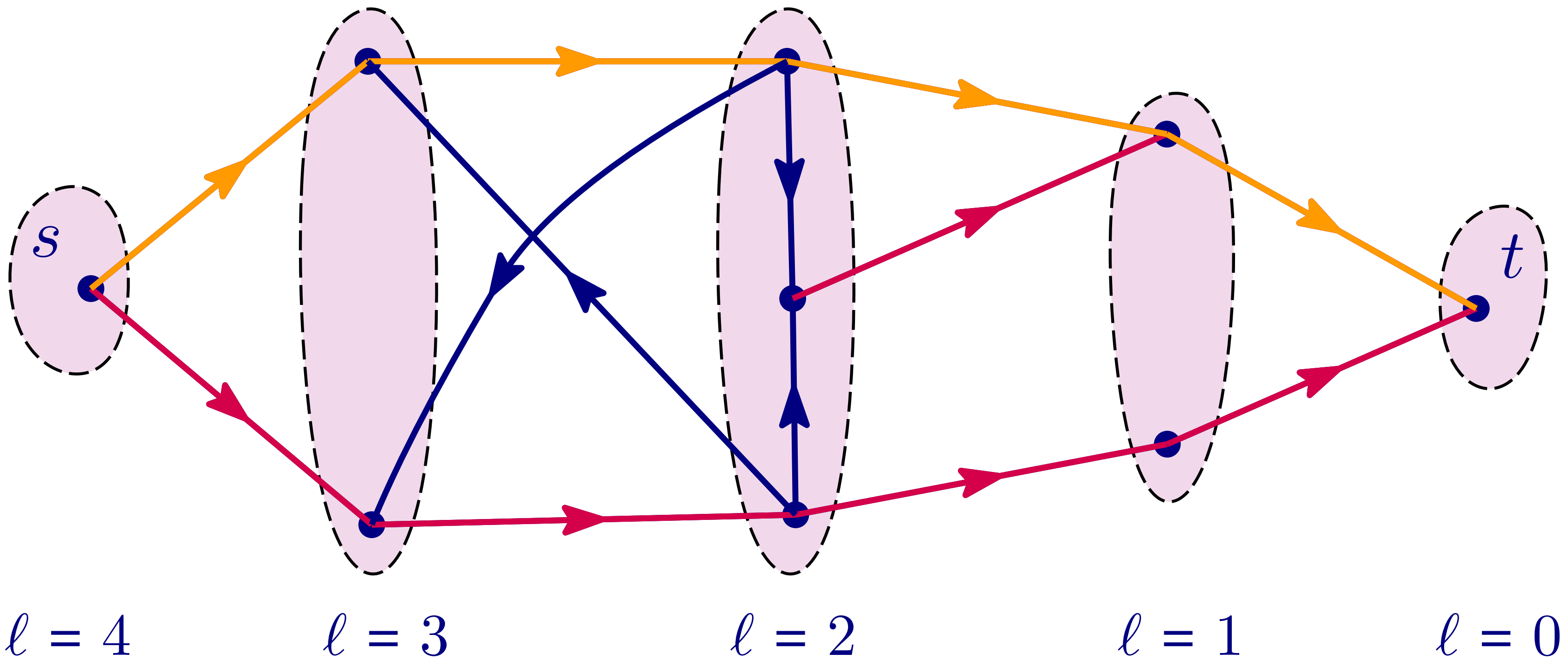

Shortest Augmenting Path: follow admissible edges from s

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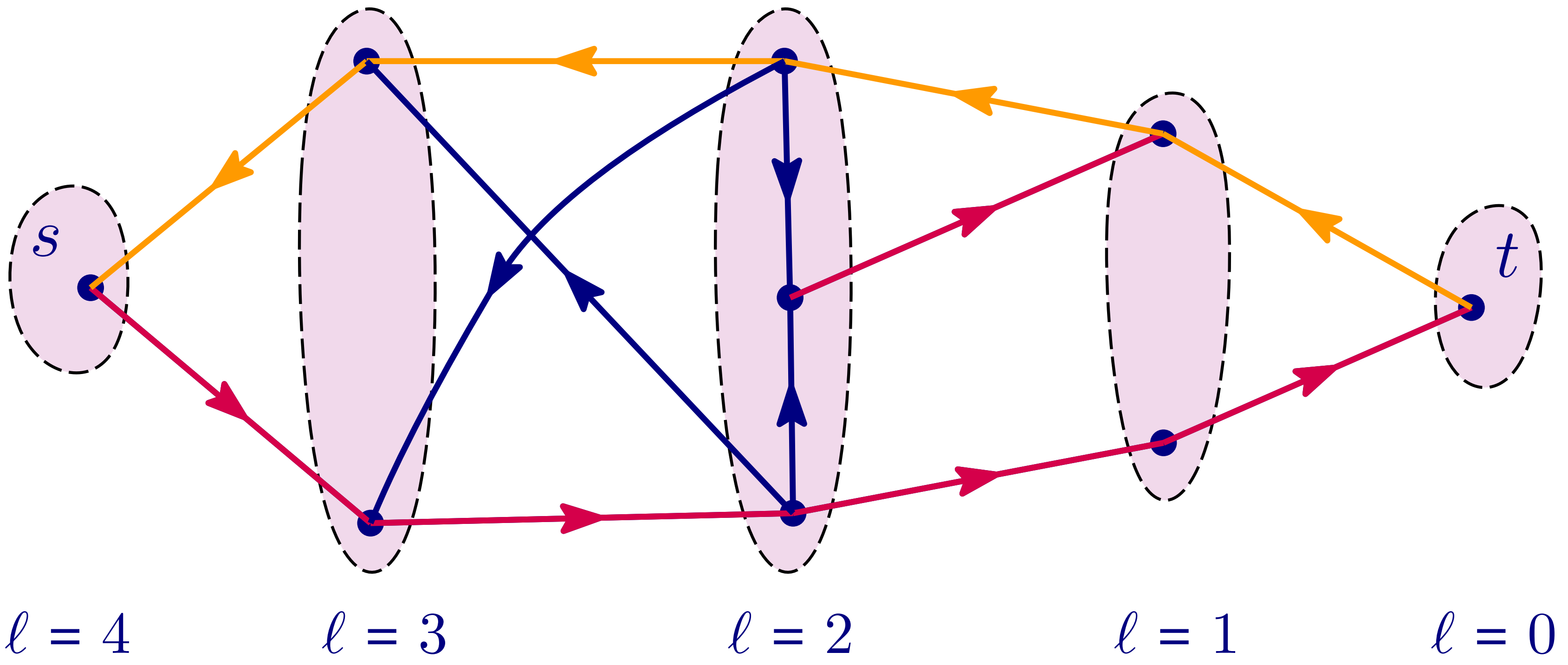

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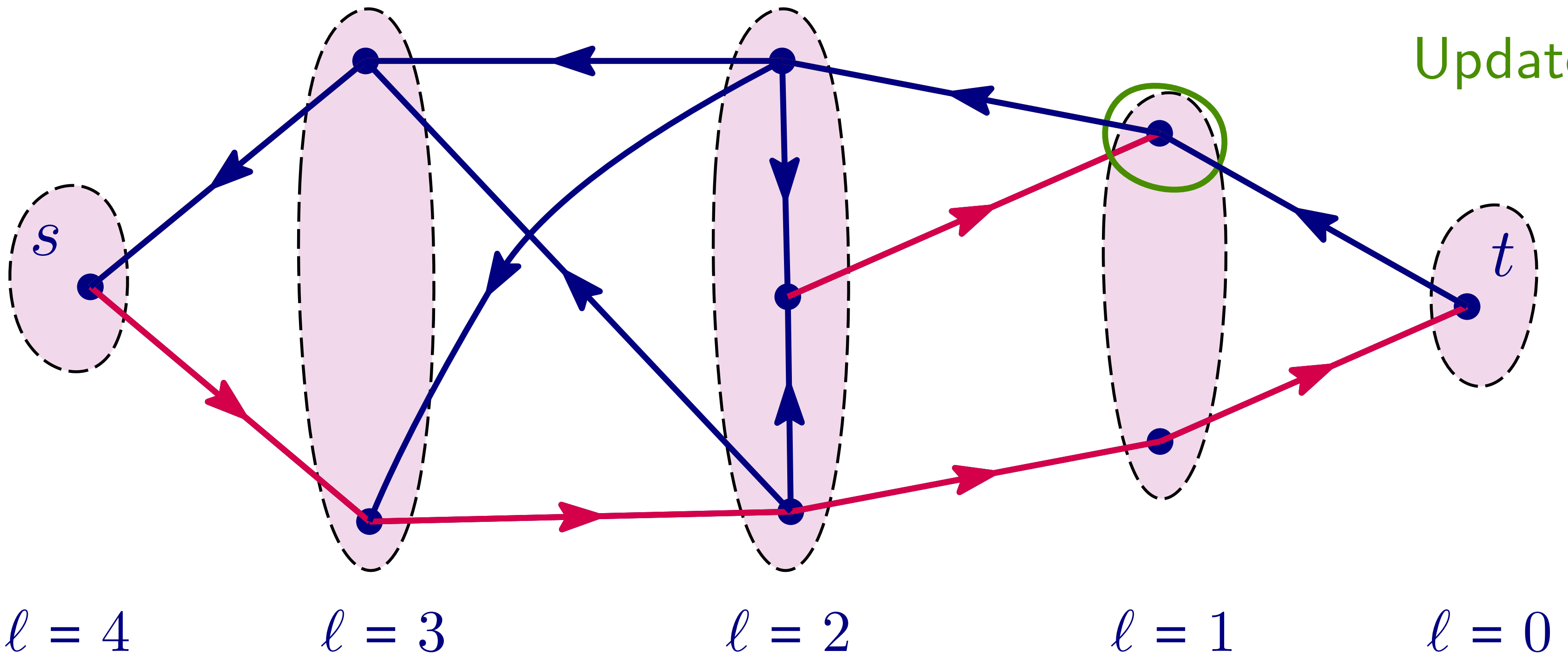
Update ℓ?

Only increases!



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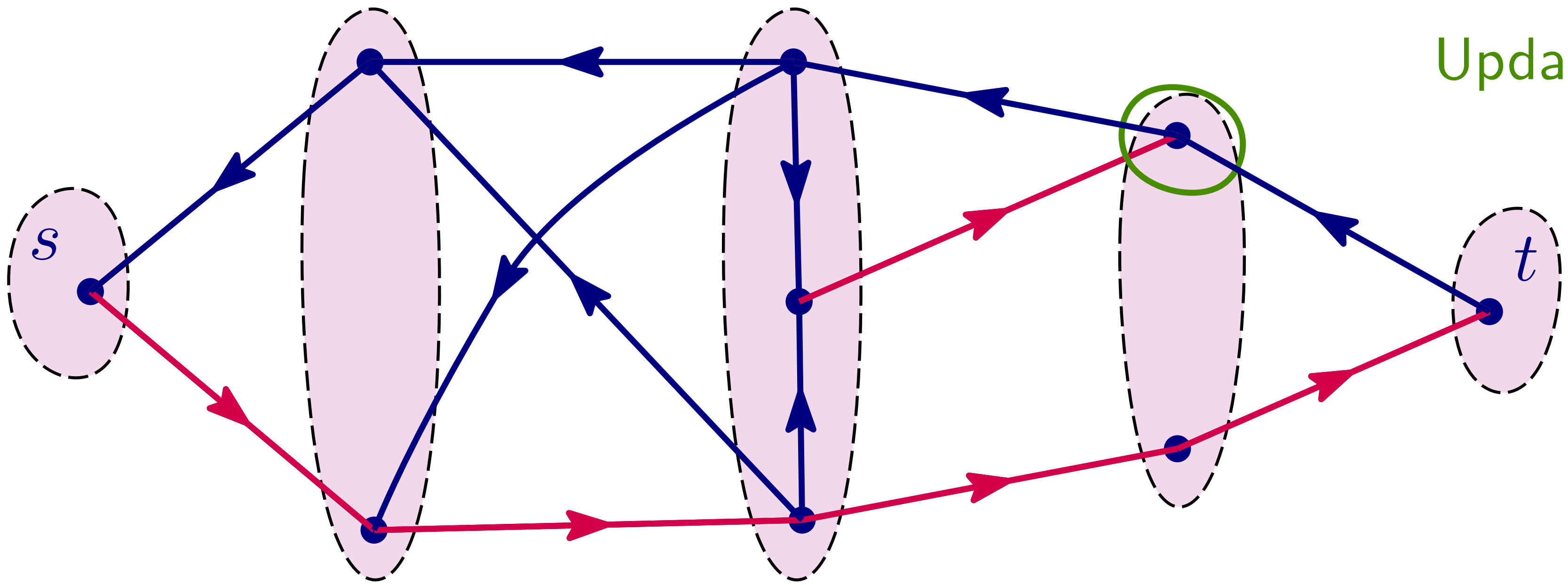

$\ell = 4$ $\ell = 3$ $\ell = 2$ $\ell = 1$ $\ell = 0$

Shortest Augmenting Path: follow admissible edges from s

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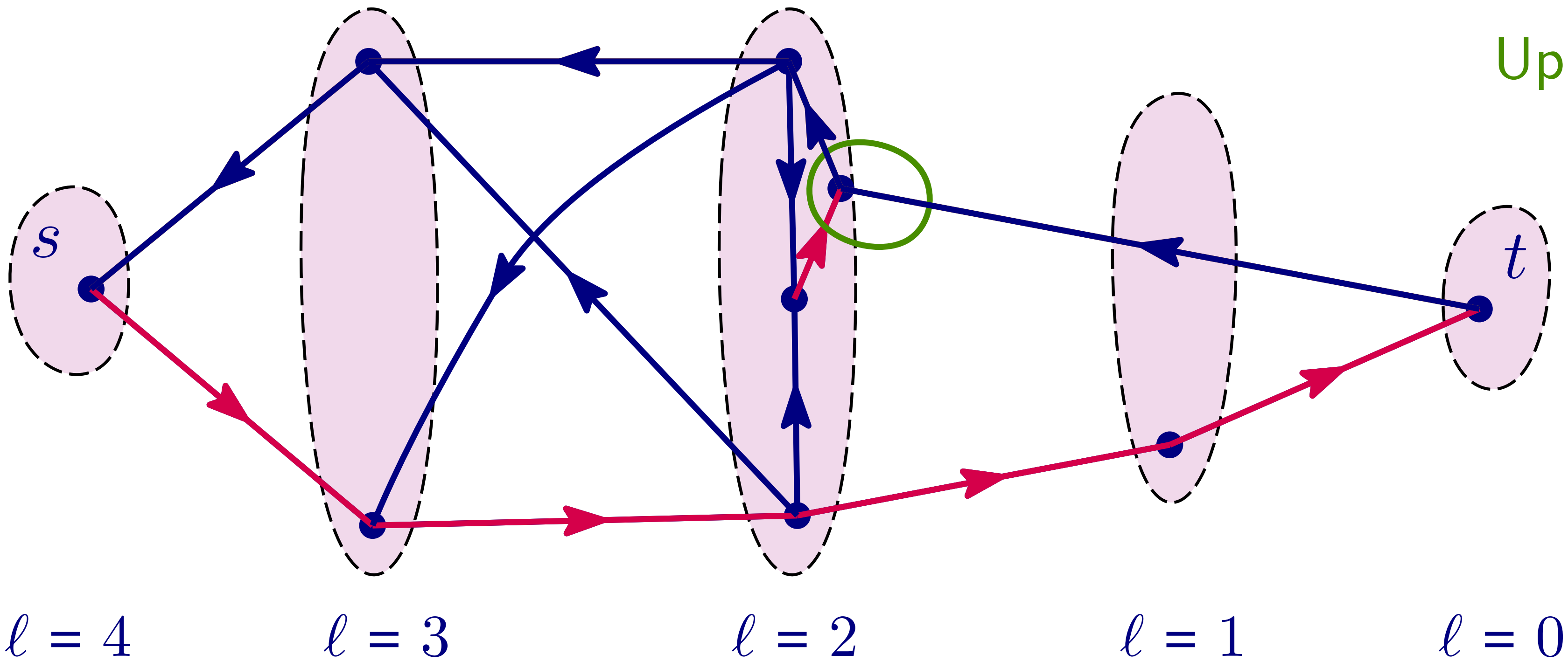

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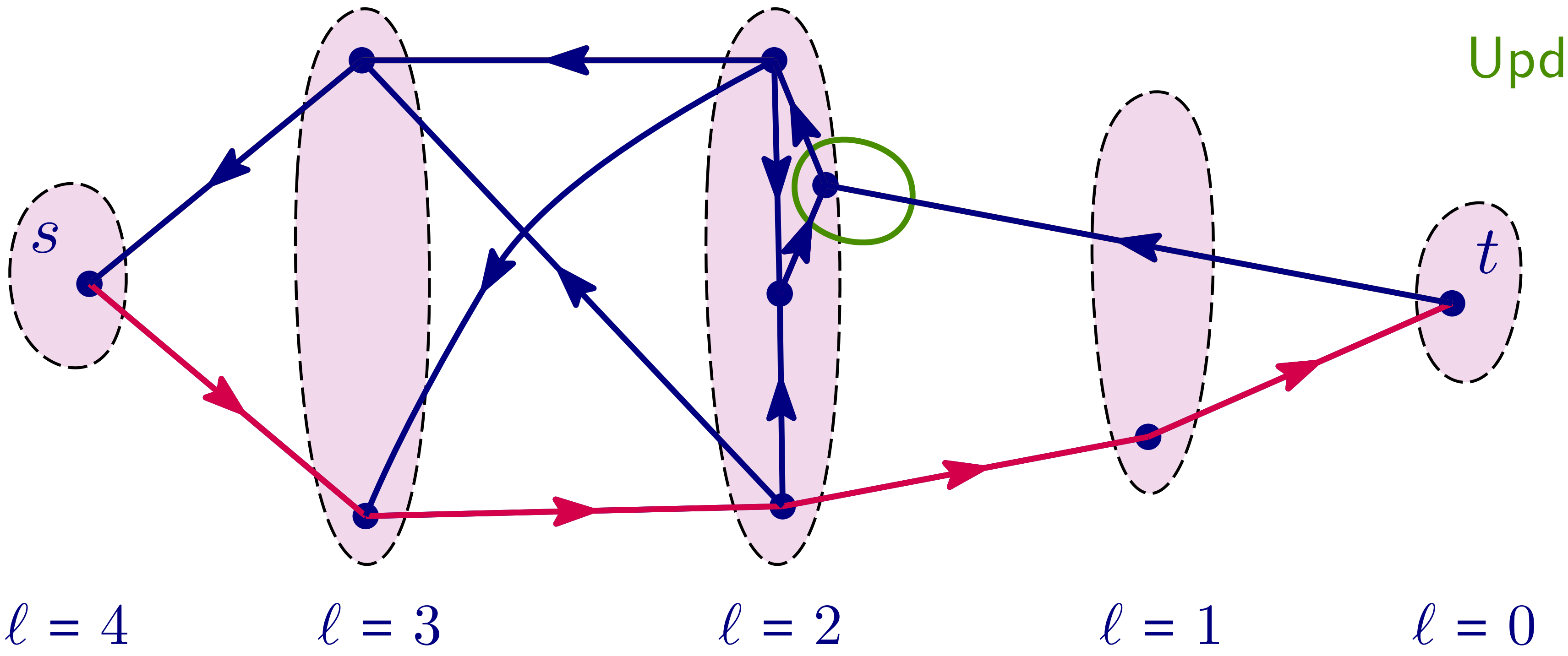

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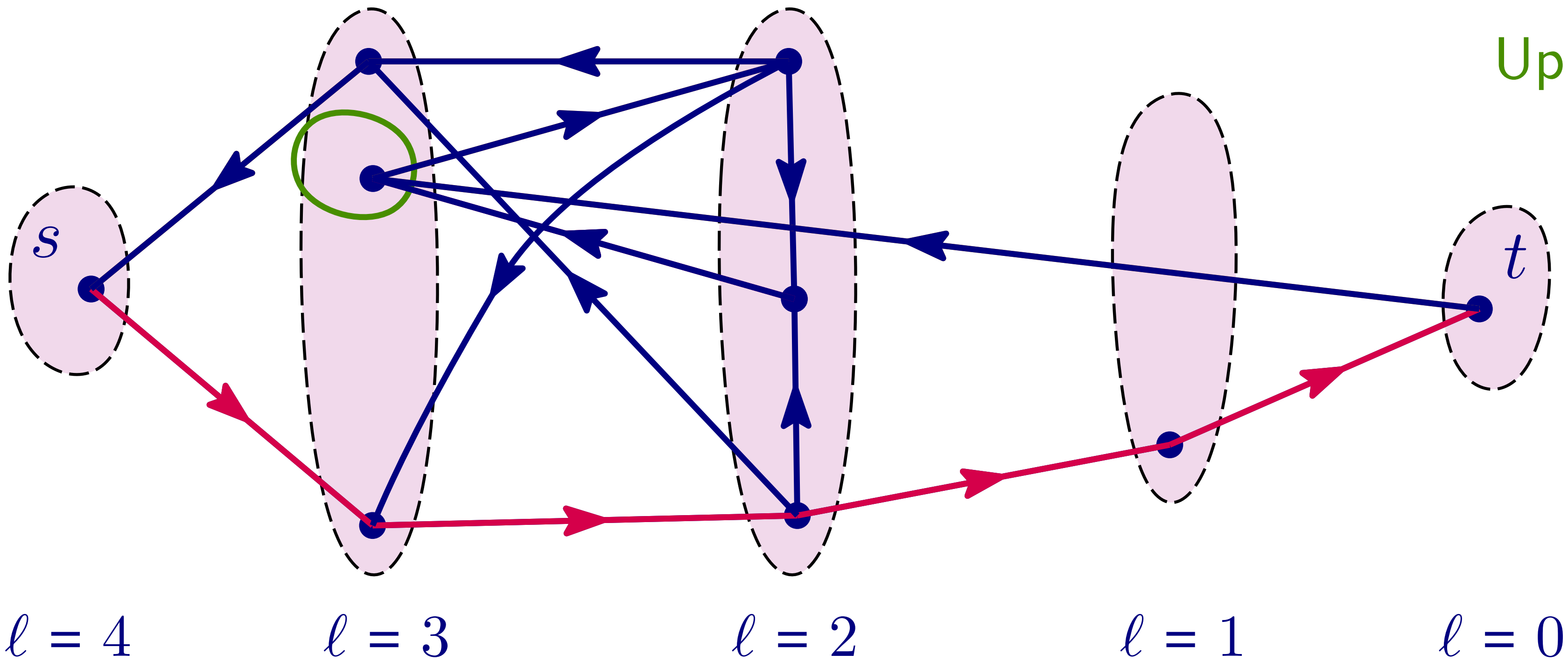

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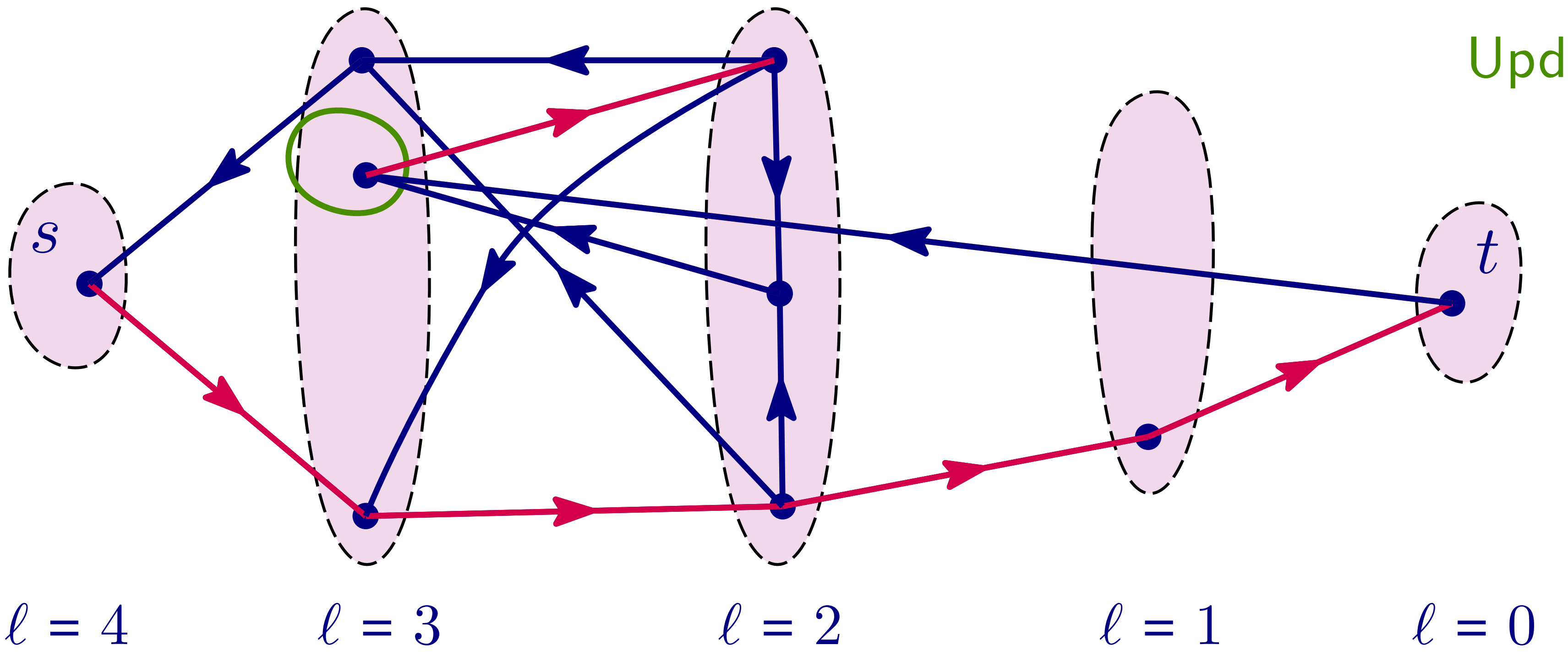

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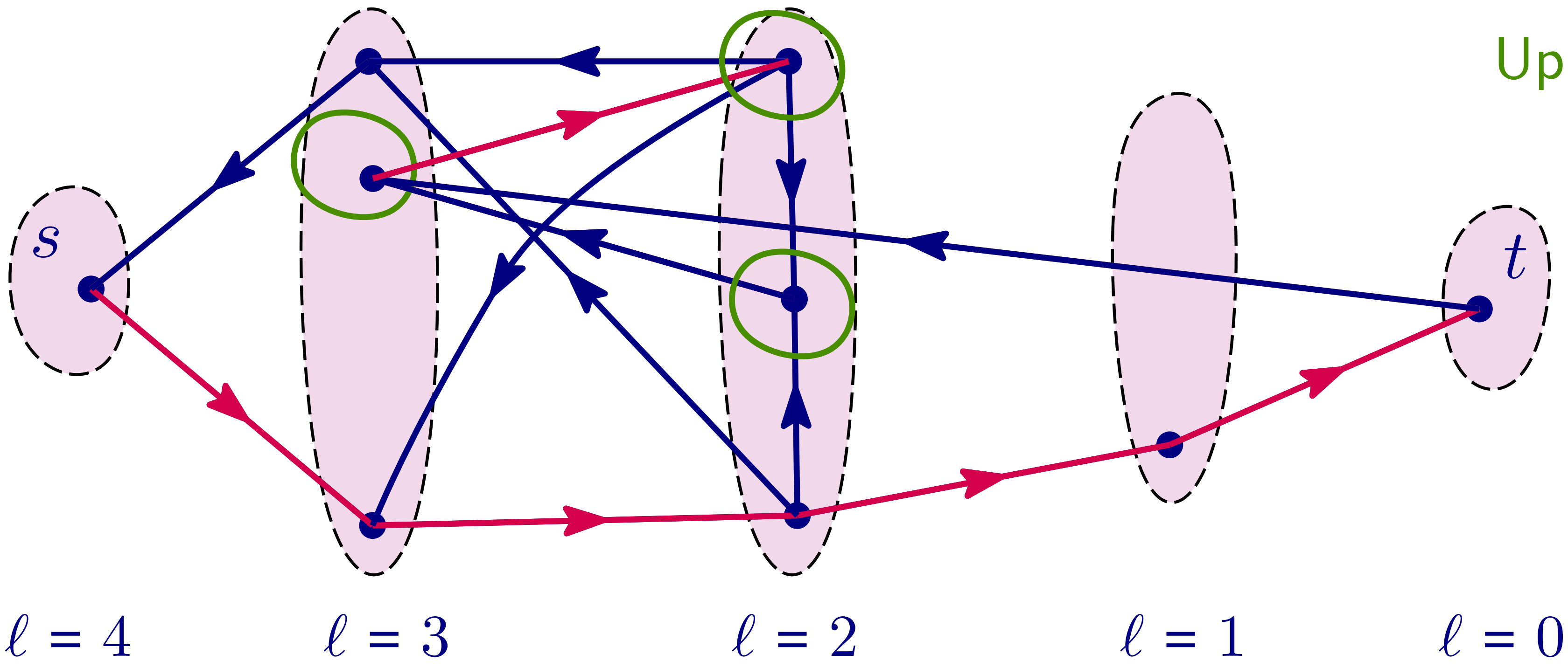

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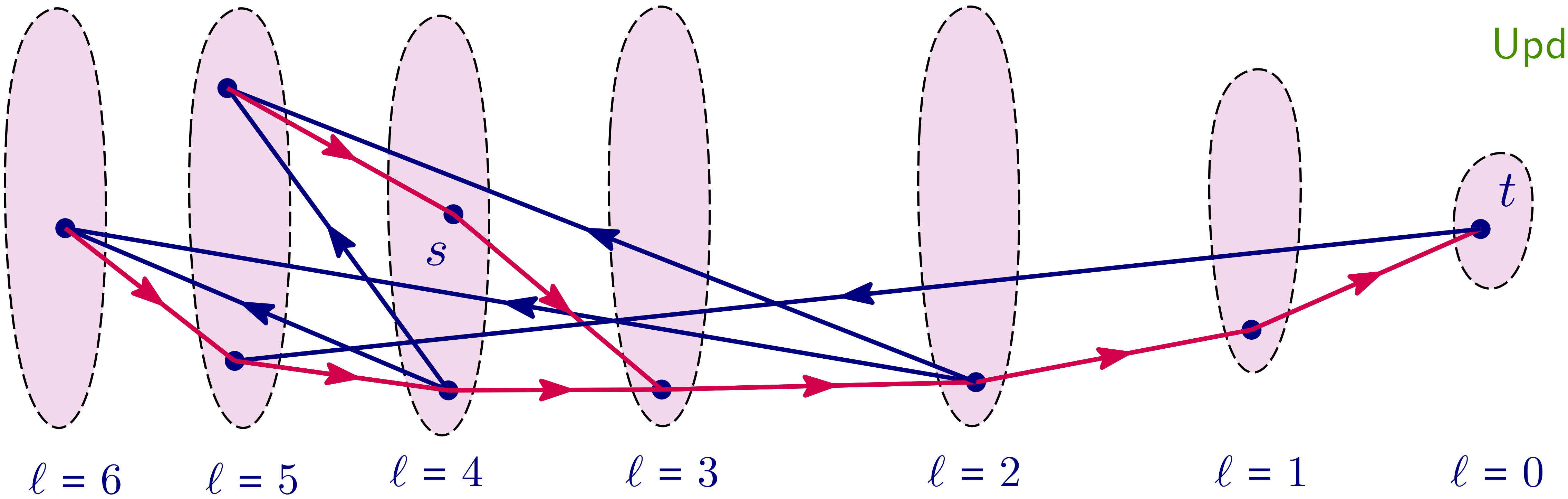

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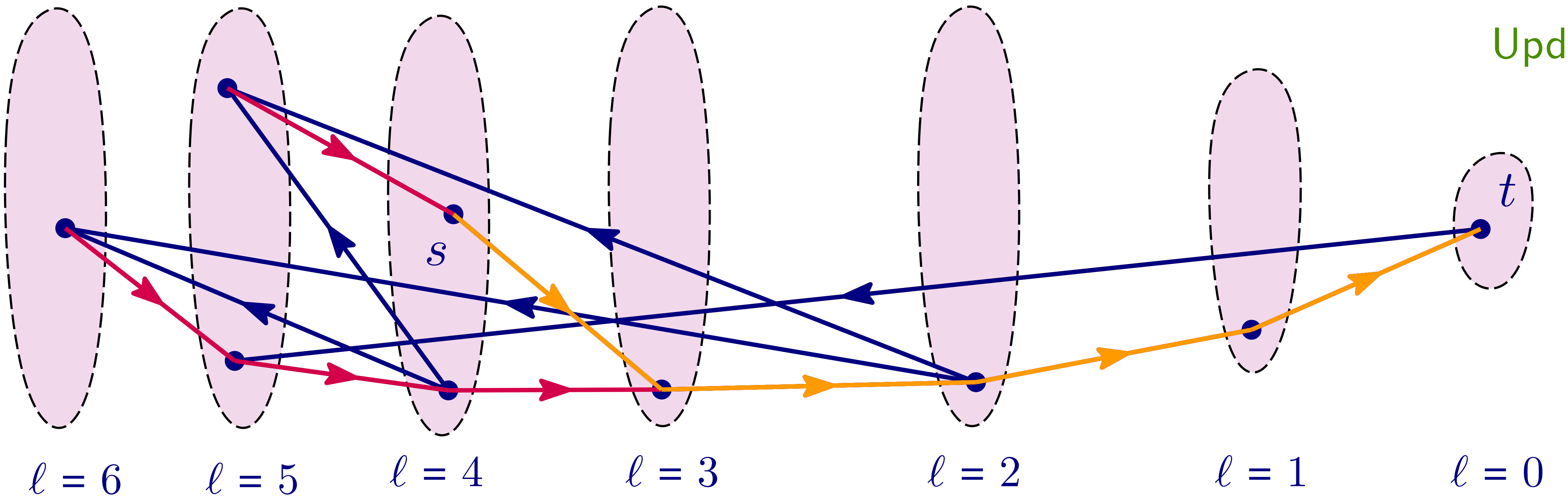

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Push-Relabel / Augment-Relabel [Goldberg-Tarjan'88] —Analysis

RELABEL $\quad O(n^2) \quad$ (n vertices, n layers)

Push-Relabel / Augment-Relabel [Goldberg-Tarjan'88] —Analysis

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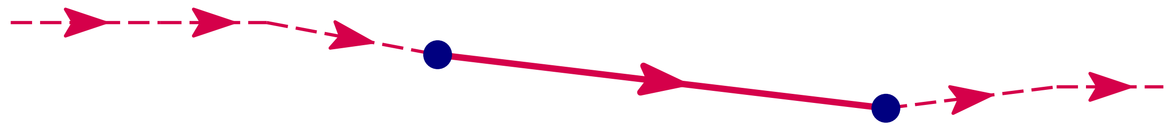

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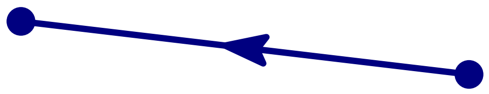

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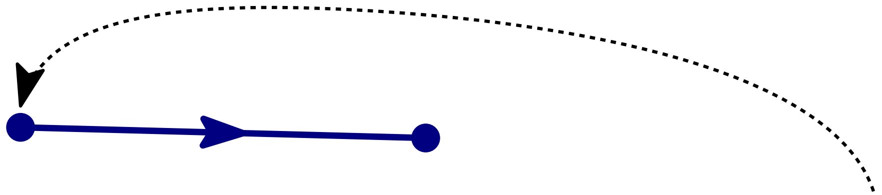

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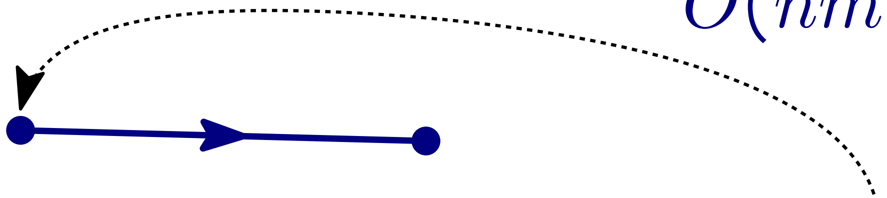

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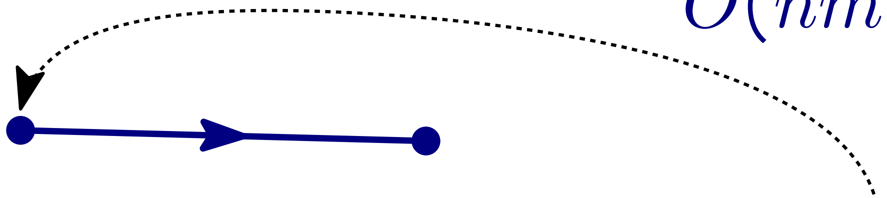

Total: $\tilde{O}(nm)$

How to speed it up?

New Idea: Edge Lengths

$\boldsymbol{w}(e) = 2$

$\boldsymbol{w}(e) = 10$

New Idea: Edge Lengths

"short" = "frequent"

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few long edges, potentially many short

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Potential Faster Algo:
look for short paths

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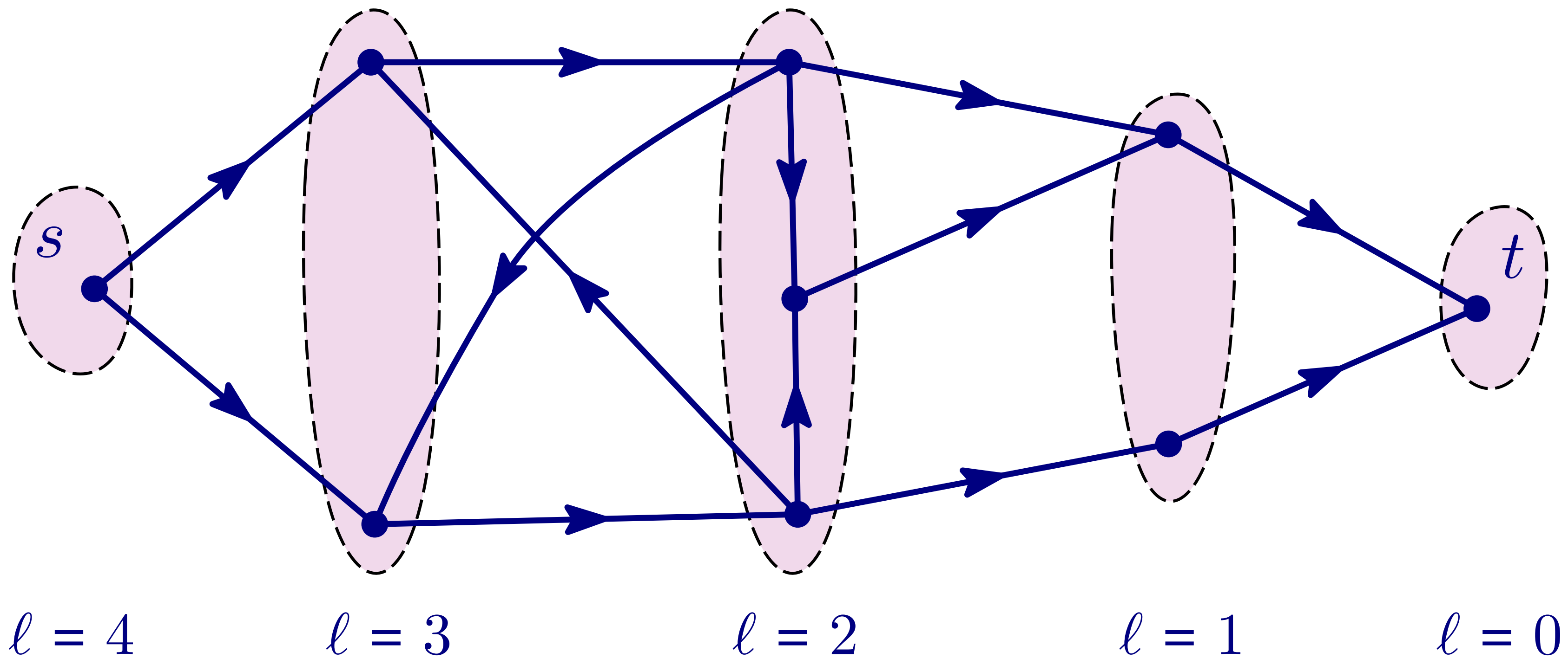
Weighted Push-Relabel

$\ell(v) = \text{dist}(v, t)$



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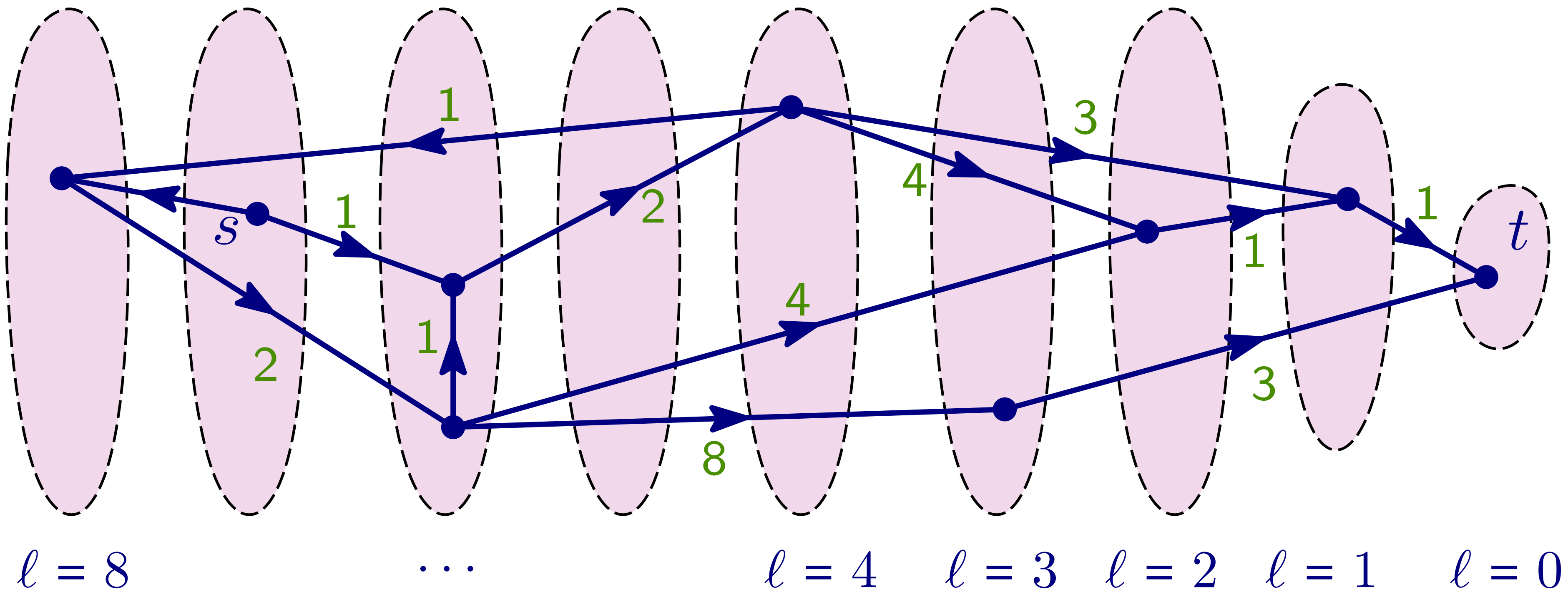

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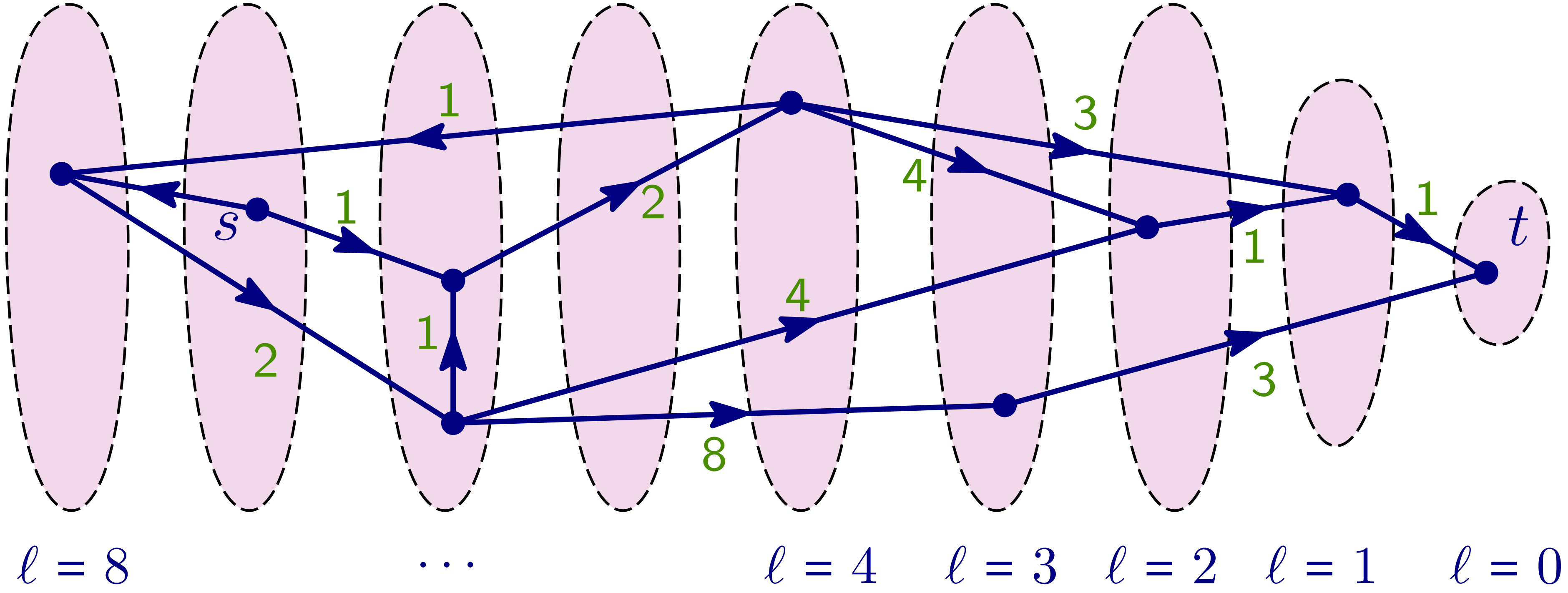
edge $e = (u, v)$ *admissible* iff $\ell(u) = \ell(v) + 1$



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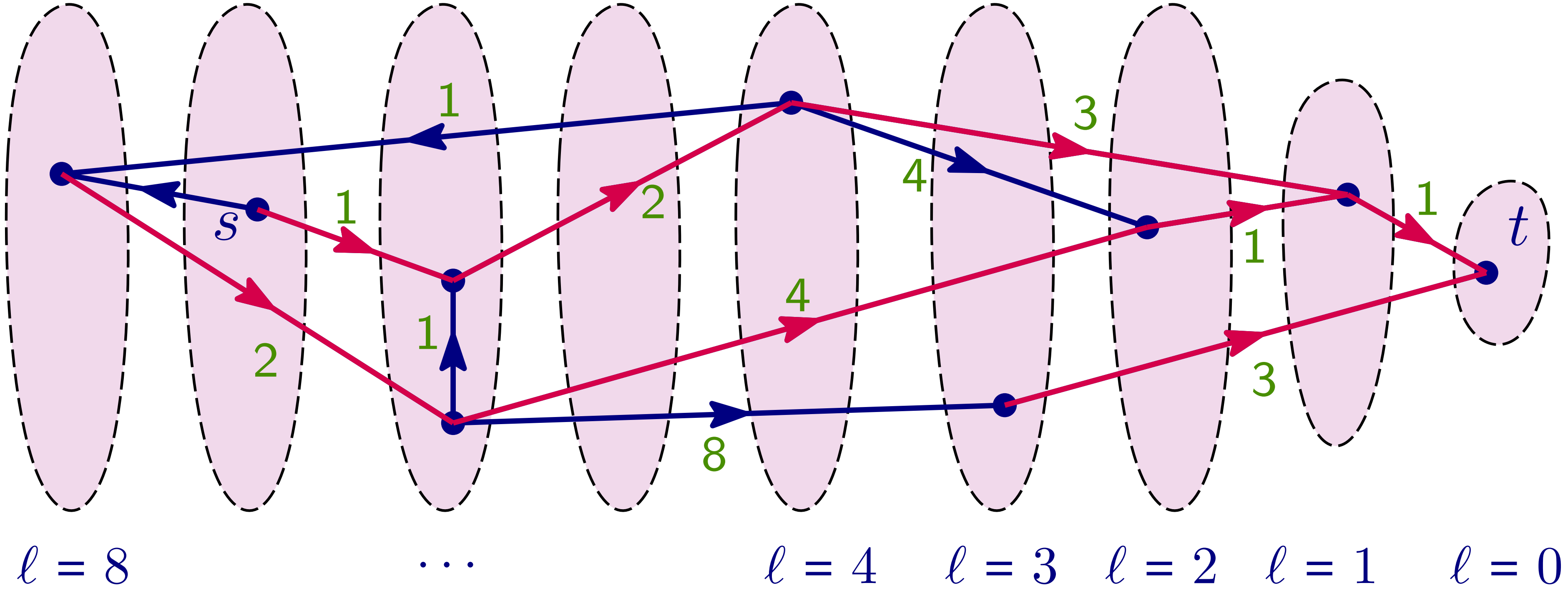

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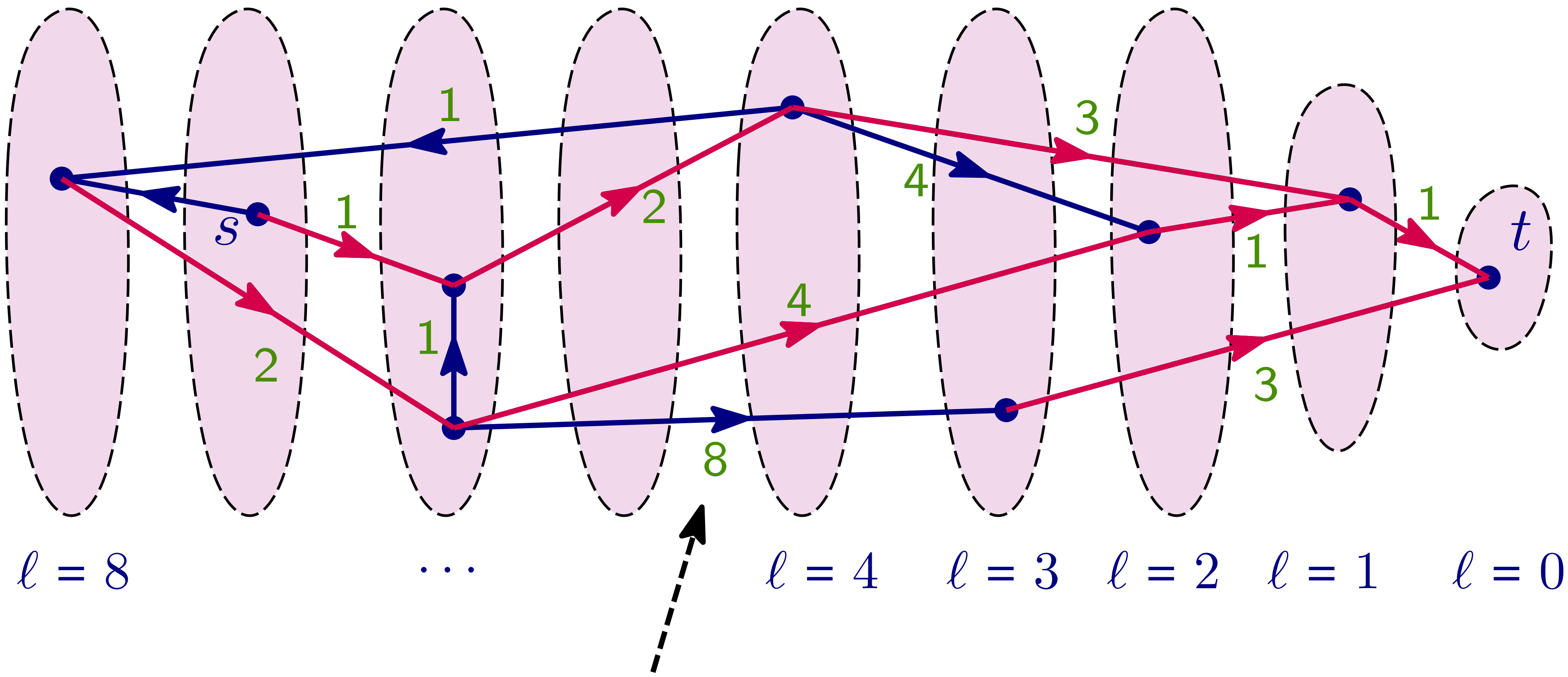

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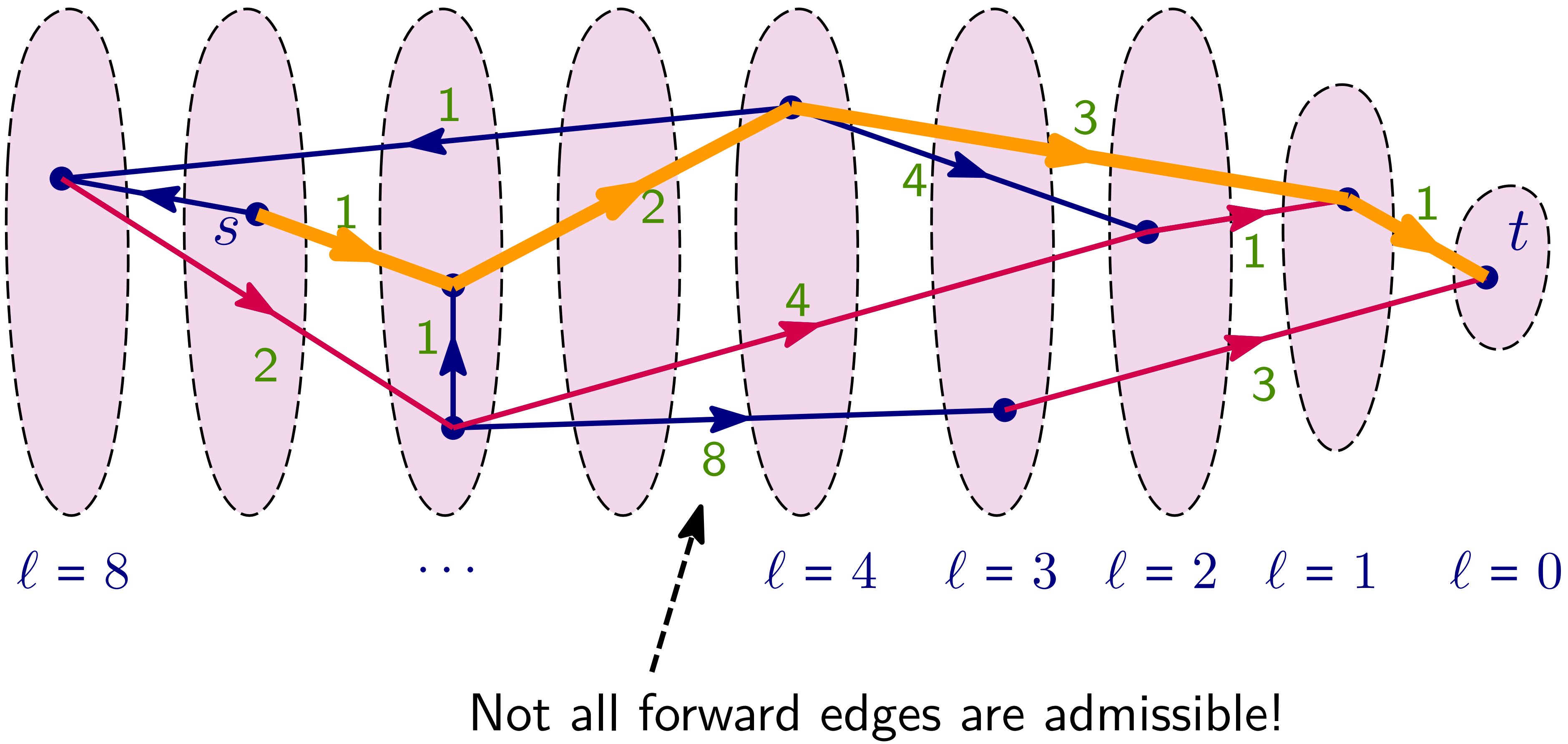

$\boldsymbol{w}$-Shortest Augmenting Path: follow admissible edges from s

Weighted Push-Relabel

$\ell(v) \approx \mathrm{dist}_{\boldsymbol{w}}(v,t)$

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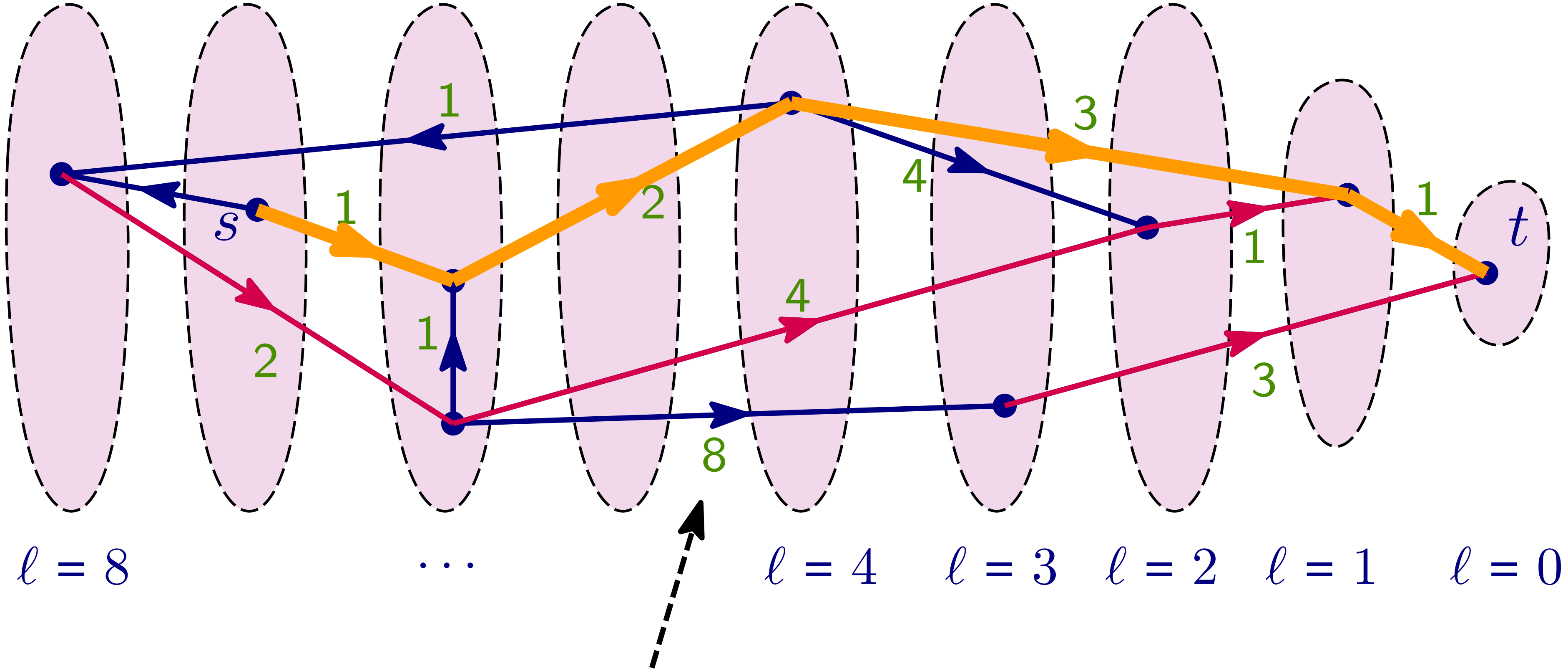

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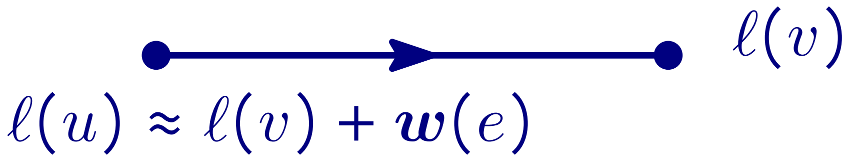

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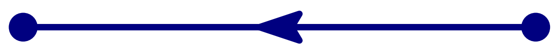

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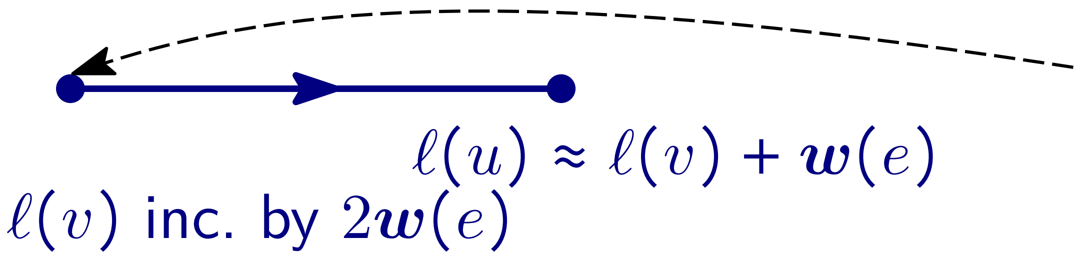


Pseudo-Code

Algorithm 1: $\textsc{PushRelabel}(G, \boldsymbol{c}, \boldsymbol{\Delta}, \boldsymbol{\nabla}, \boldsymbol{w}, h)$

Initialize $\boldsymbol{f}$ as the empty flow.
Let $\boldsymbol{\ell}(v) = 0$ for all $v \in V$. `// levels`
Mark each edge $e \in \overrightarrow{E} \cup \overleftarrow{E}$ as *inadmissible* and all vertices as *alive*.

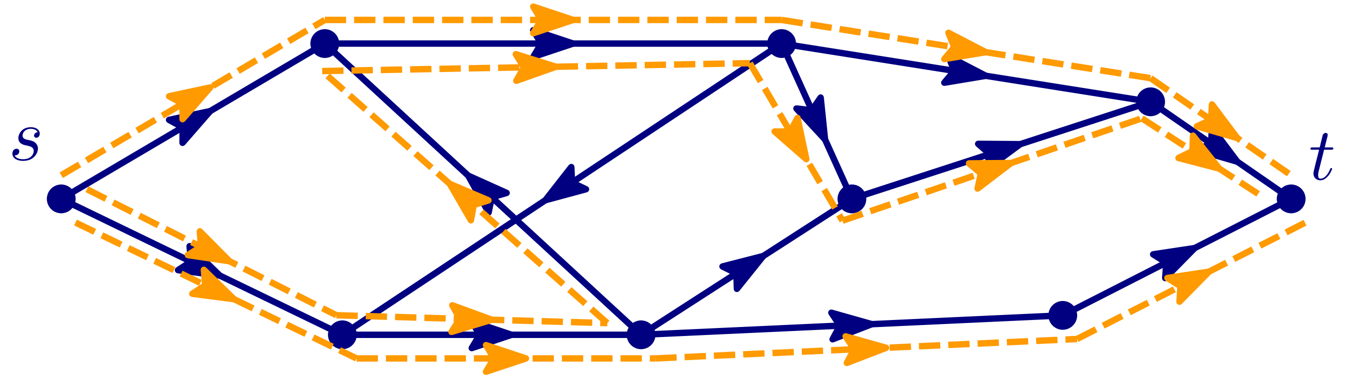
function $\textsc{Relabel}(v)$
  Set $\boldsymbol{\ell}(v) \leftarrow \boldsymbol{\ell}(v) + 1$.
  if $\boldsymbol{\ell}(v) > 9h$ **then**
    mark v as *dead* and **return**.
  for *each edge* $e \ni v$ *where* $\boldsymbol{w}(e)$ *divides* $\boldsymbol{\ell}(v)$ **do**
    Let $(x, y) = e$.
    if $\boldsymbol{\ell}(x) - \boldsymbol{\ell}(y) \geq 2\boldsymbol{w}(e)$ *and* $\boldsymbol{c_f}(e) > 0$ **then** mark e as *admissible*.
    else mark e as *inadmissible*.

main loop
  while *there is an alive vertex* v *with* $\boldsymbol{\nabla_f}(v) = 0$ *and without an admissible out-edge* **do**
    $\textsc{Relabel}(v)$
  if *there is some alive vertex* s *with* $\boldsymbol{\Delta_f}(s) > 0$ **then**
    `// P is an "augmenting path"`
    Trace a path P from s to some sink t, by arbitrarily following admissible out-edges.
    Let $c^{\text{augment}} \leftarrow \min\{\boldsymbol{\Delta_f}(s), \boldsymbol{\nabla_f}(t), \min_{e \in P} \boldsymbol{c_f}(e)\}$.
    for $e \in P$ **do** `// Augment f along P`
      if *e is a forward edge* **then** $\boldsymbol{f}(e) \leftarrow \boldsymbol{f}(e) + c^{\text{augment}}$.
      else $\boldsymbol{f}(e') \leftarrow \boldsymbol{f}(e') - c^{\text{augment}}$, where e' is the corresponding forward edge to e.
      Adjust residual capacities $\boldsymbol{c_f}$ of e and the corresponding reverse edge.
      if $\boldsymbol{c_f}(e) = 0$ **then** mark e as *inadmissible*.
    `//` $\boldsymbol{\Delta_f}(s)$ `and` $\boldsymbol{\nabla_f}(t)$ `goes down by` c^{augment}
  else return $\boldsymbol{f}$

Similar to normal Augment-Relabel / Push-Relabel

Good Edge Lengths

- *Good* $\boldsymbol{w}$:
  - $\sum_{e \in E} \frac{n}{\boldsymbol{w}(e)}$ is small $(\approx n^{2+o(1)}$, running-time)
  - "Optimal" flow $f^\star$ which is short w.r.t. $\boldsymbol{w}$ (flow paths of length $\approx n^{1+o(1)}$)



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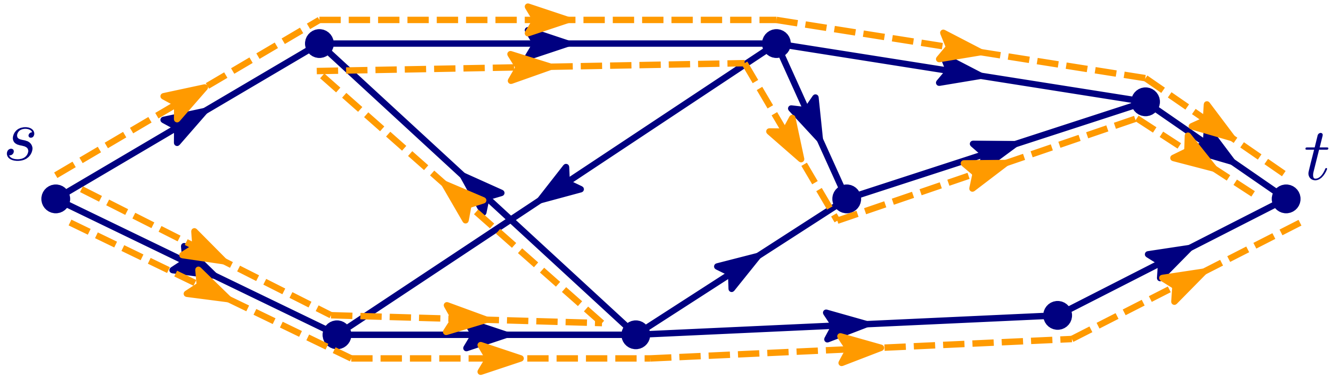
Lemma.

Weighted Push-Relabel finds $\boldsymbol{f}$ with $|\boldsymbol{f}| \geq \frac{1}{10}|\boldsymbol{f}^\star|$



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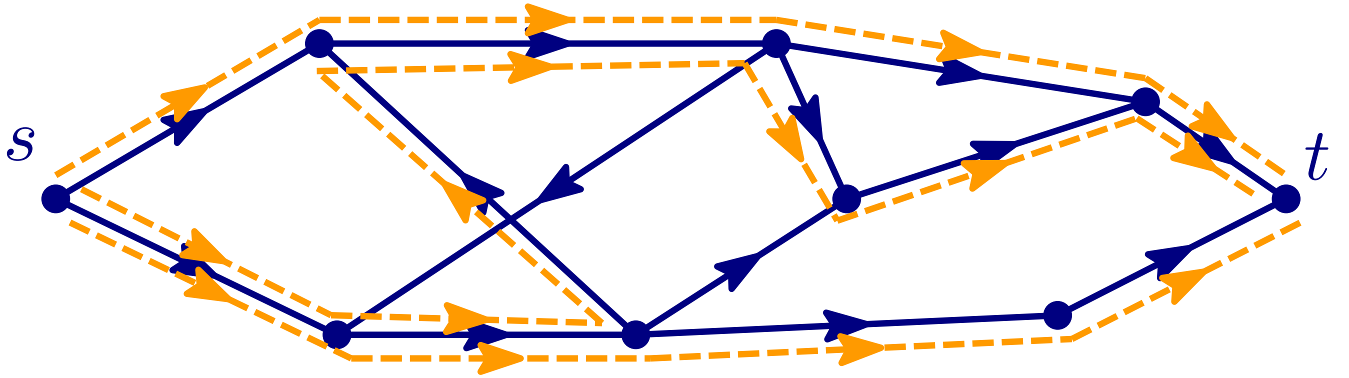
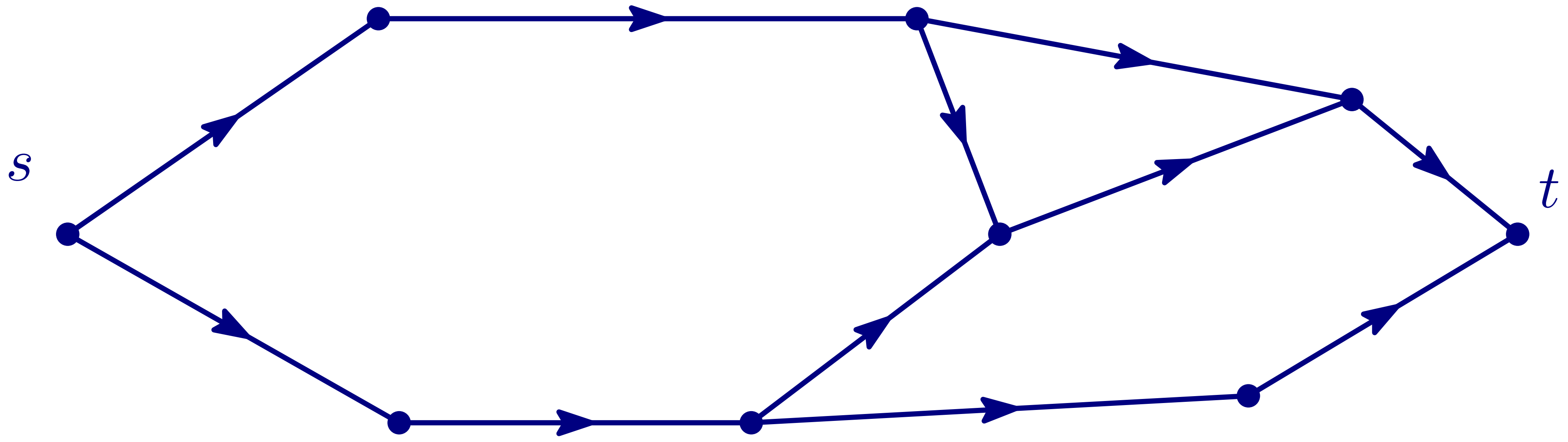


Proof Sketch.
If not: $|\boldsymbol{f}| < \frac{1}{10}|\boldsymbol{f}^\star|$
$\implies$ some flow path is still short in residual graph $G_{\boldsymbol{f}}$

How to find good edge lengths?

Directed Acyclic Graphs (DAG)

Def: no directed cycles



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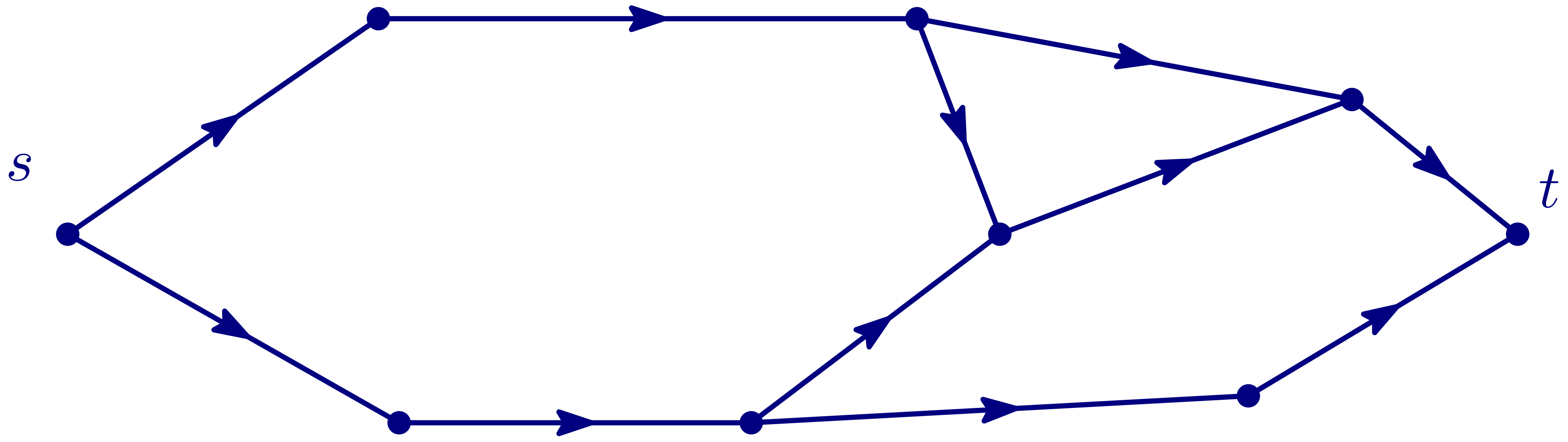

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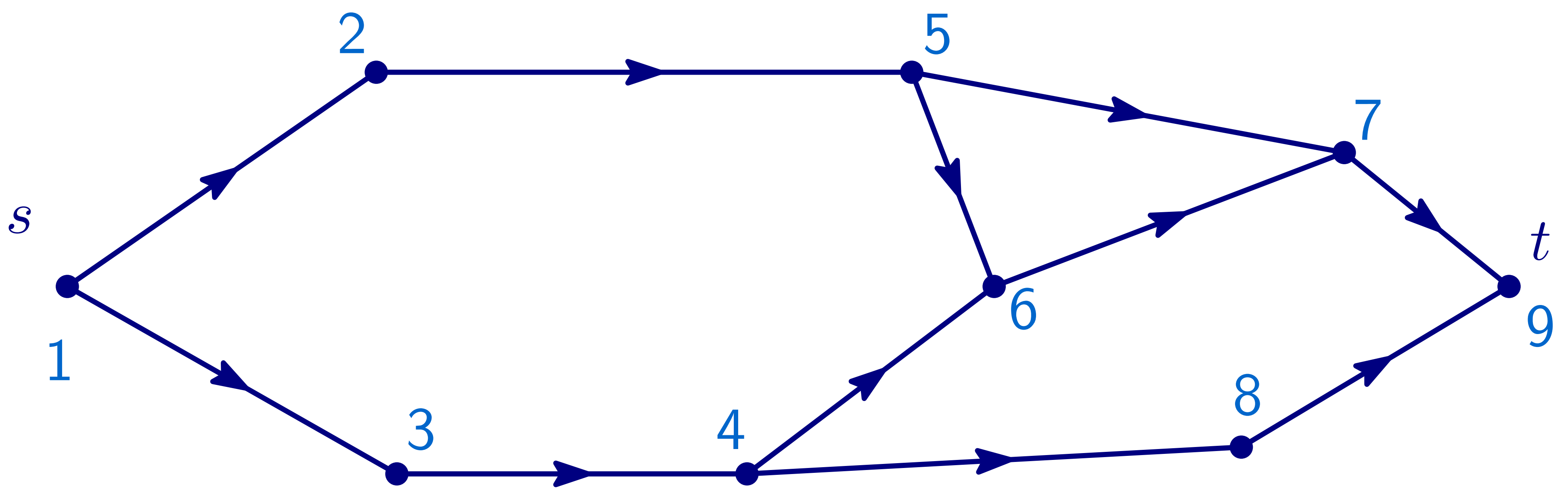

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$$\boldsymbol{w}(\boldsymbol{u},\boldsymbol{v}) = |\boldsymbol{\tau}(u) - \boldsymbol{\tau}(v)|$$

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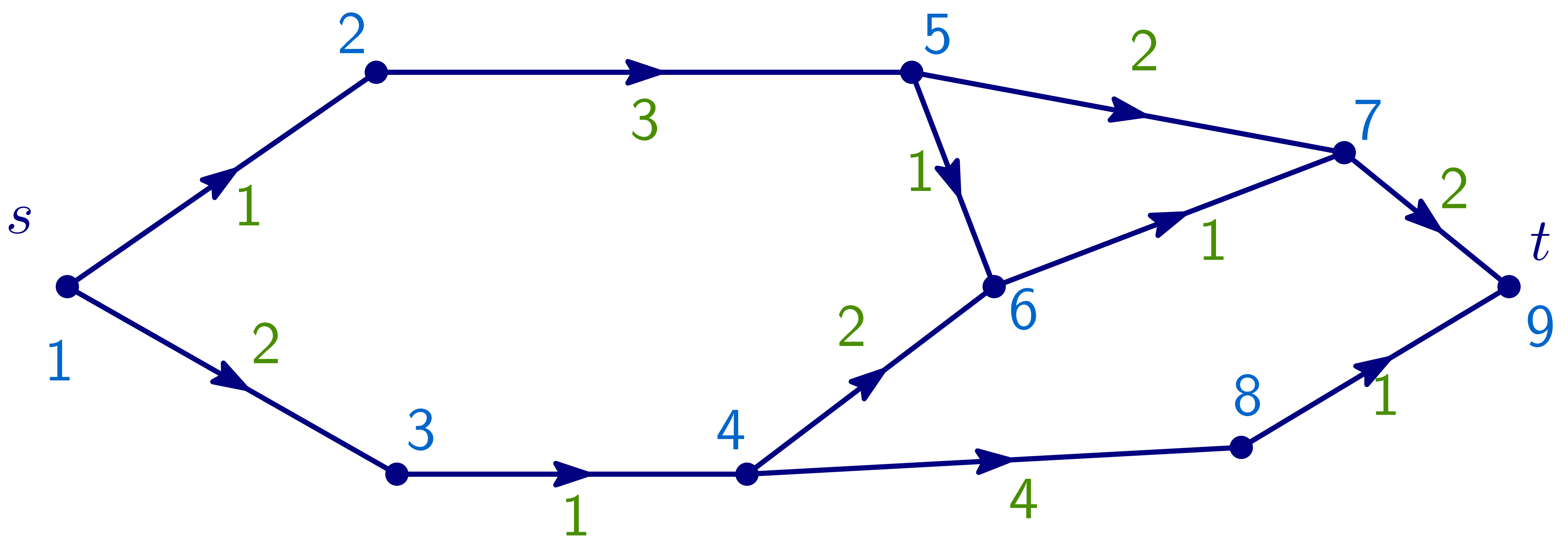

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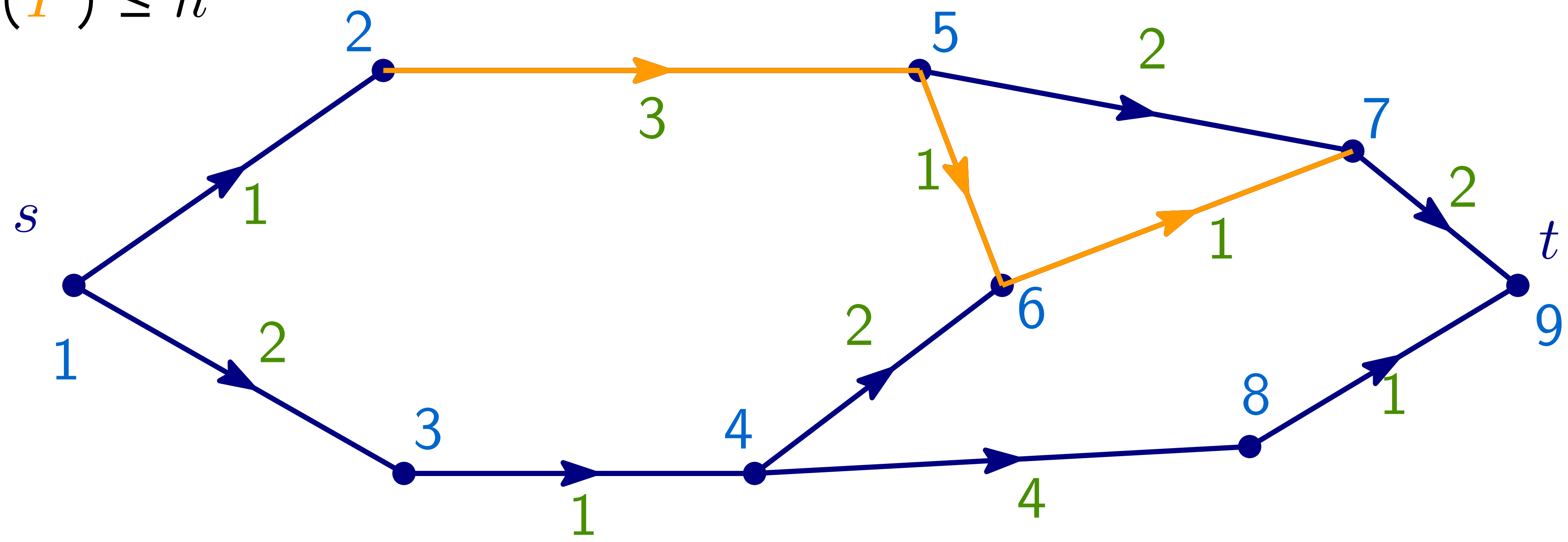

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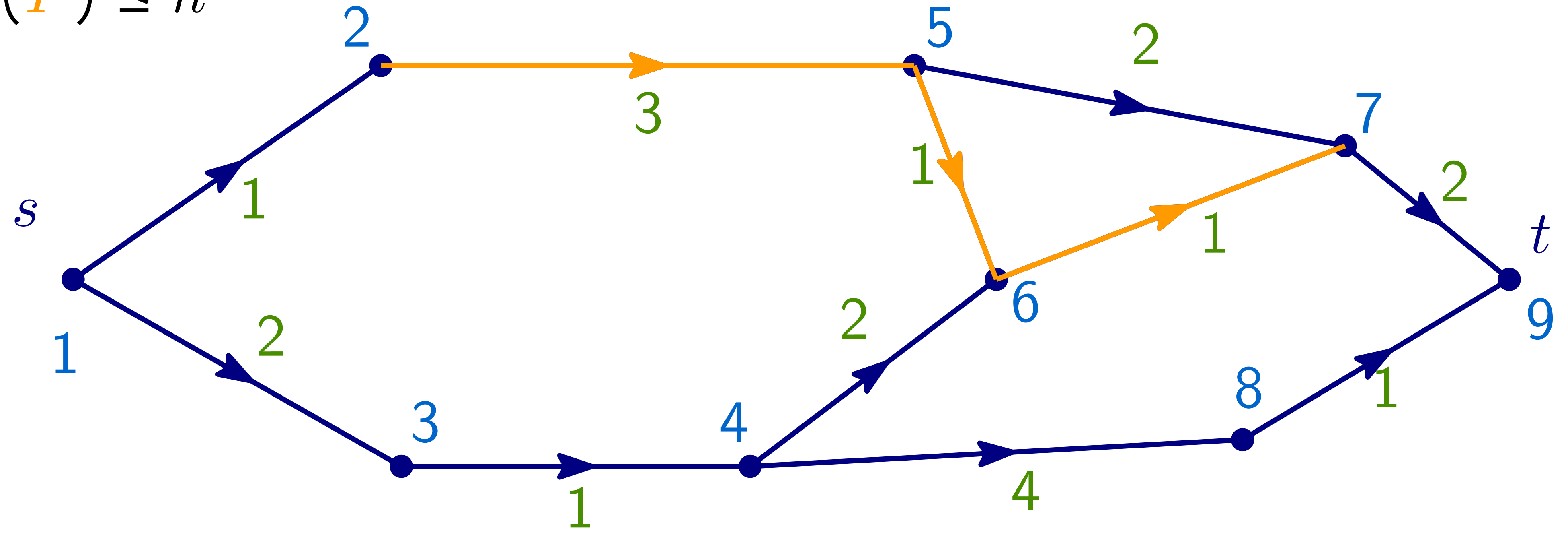

$\sum_{(u,v)} \frac{n}{|\boldsymbol{\tau}(u) - \boldsymbol{\tau}(v)|} \leq n \sum_{v \in V} \sum_{k=1}^{n-1} \frac{1}{k} \leq n^2 \log n$

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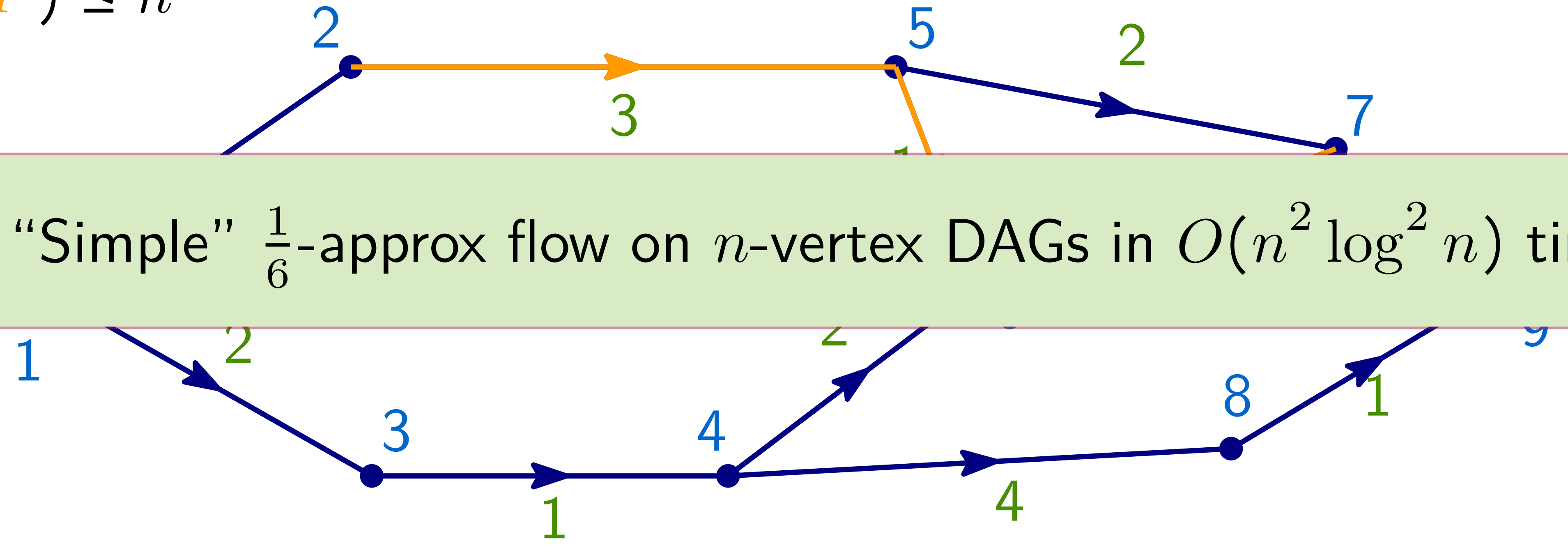

Theorem: "Simple" $\frac{1}{6}$-approx flow on n-vertex DAGs in $O(n^2 \log^2 n)$ time.

$$\sum_{(u,v)} \frac{n}{|\boldsymbol{\tau}(u) - \boldsymbol{\tau}(v)|} \le n \sum_{v \in V} \sum_{k=1}^{n-1} \frac{1}{k} \le n^2 \log n$$

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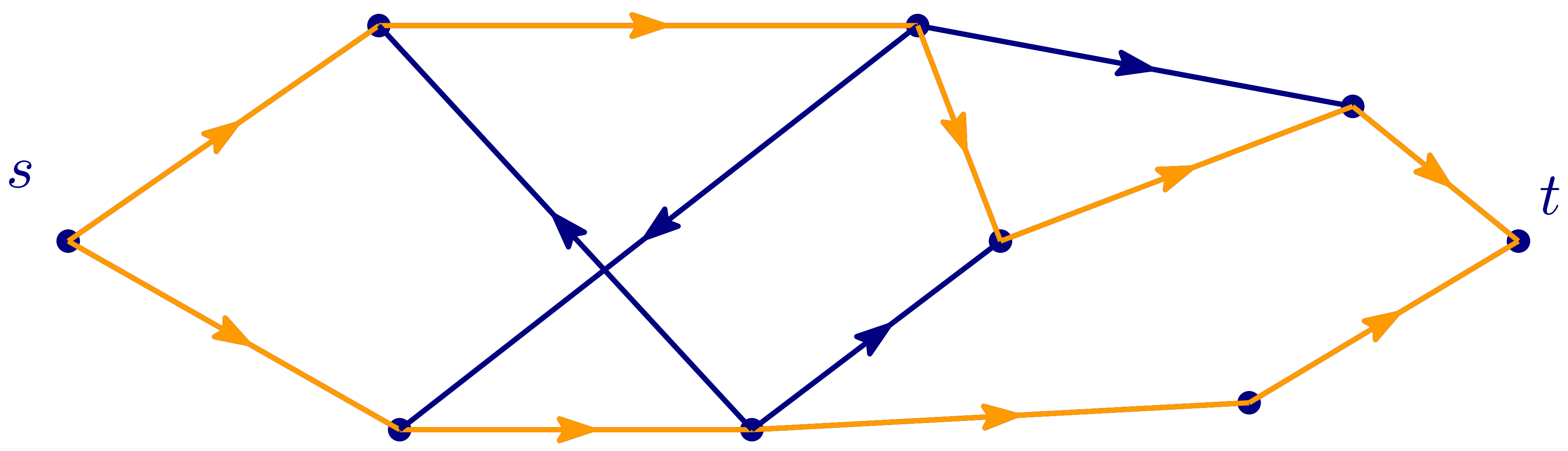

General Graphs — Attempt One

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2. Look at graph induced by $f^\star$

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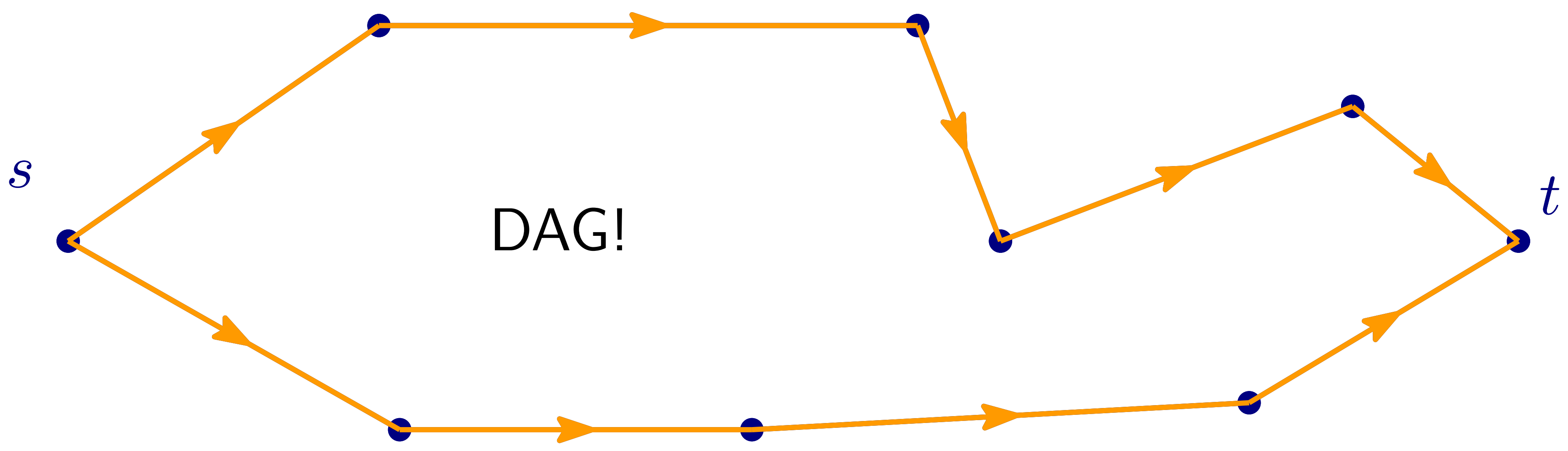

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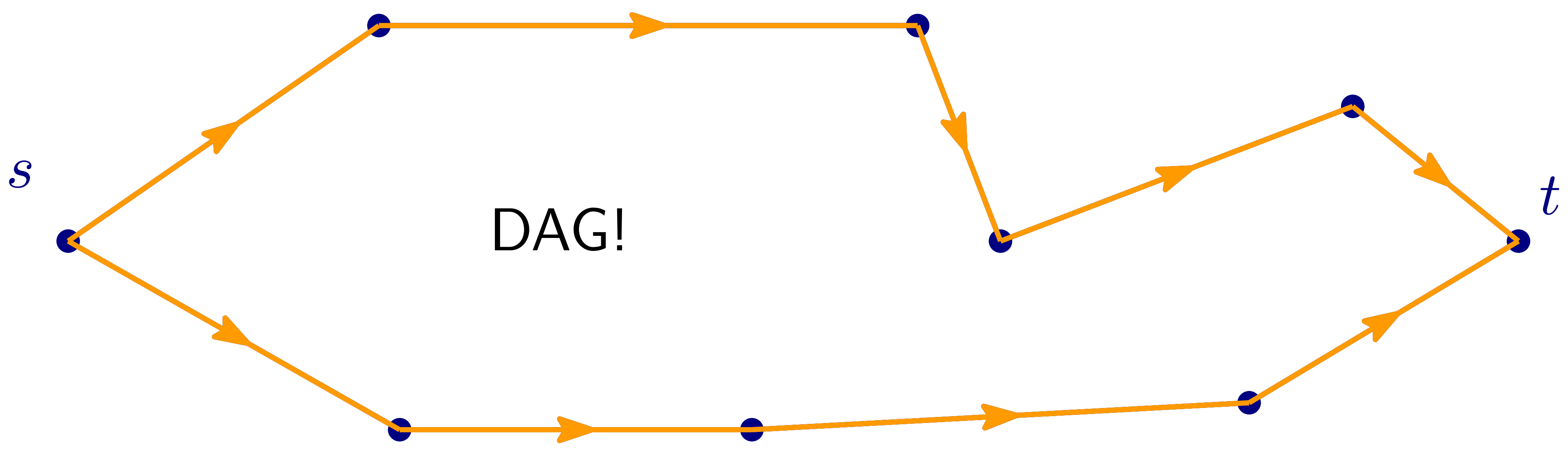

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4. Use weighted push-relabel to solve approx maxflow :)

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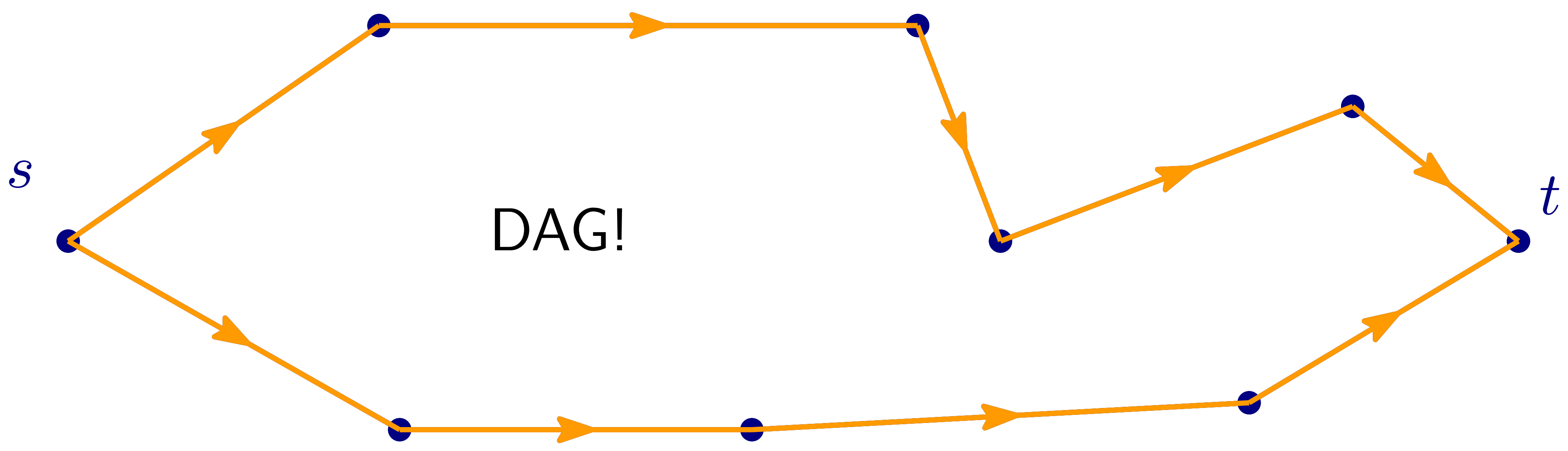

General Graphs — Attempt One

1. Compute maxflow $f^\star$ ←**Cheating!**
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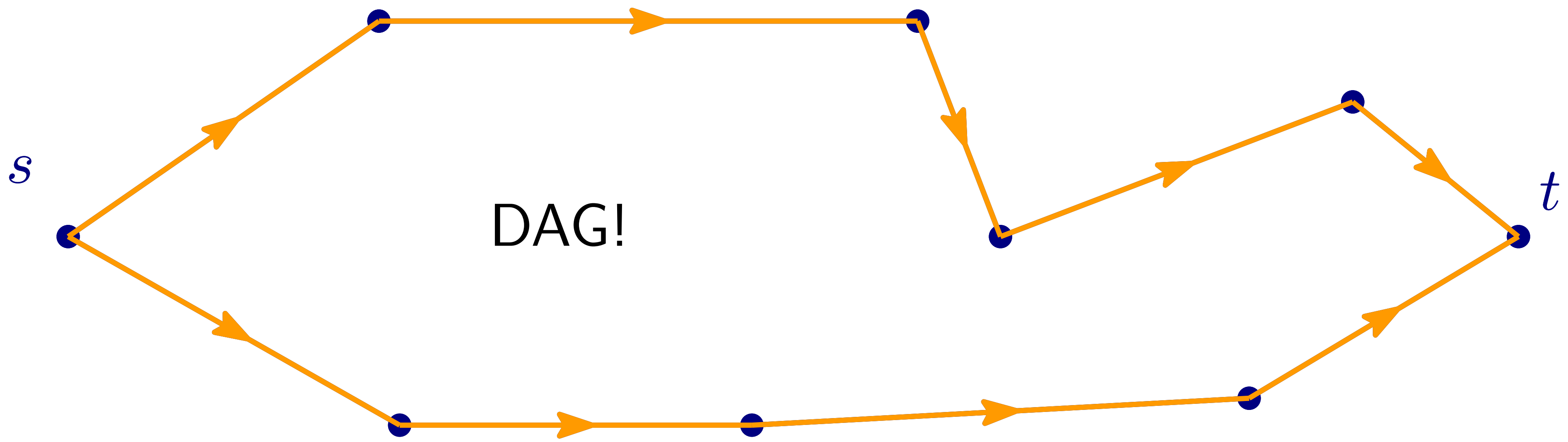

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Good edge lenghts $\boldsymbol{w}$?

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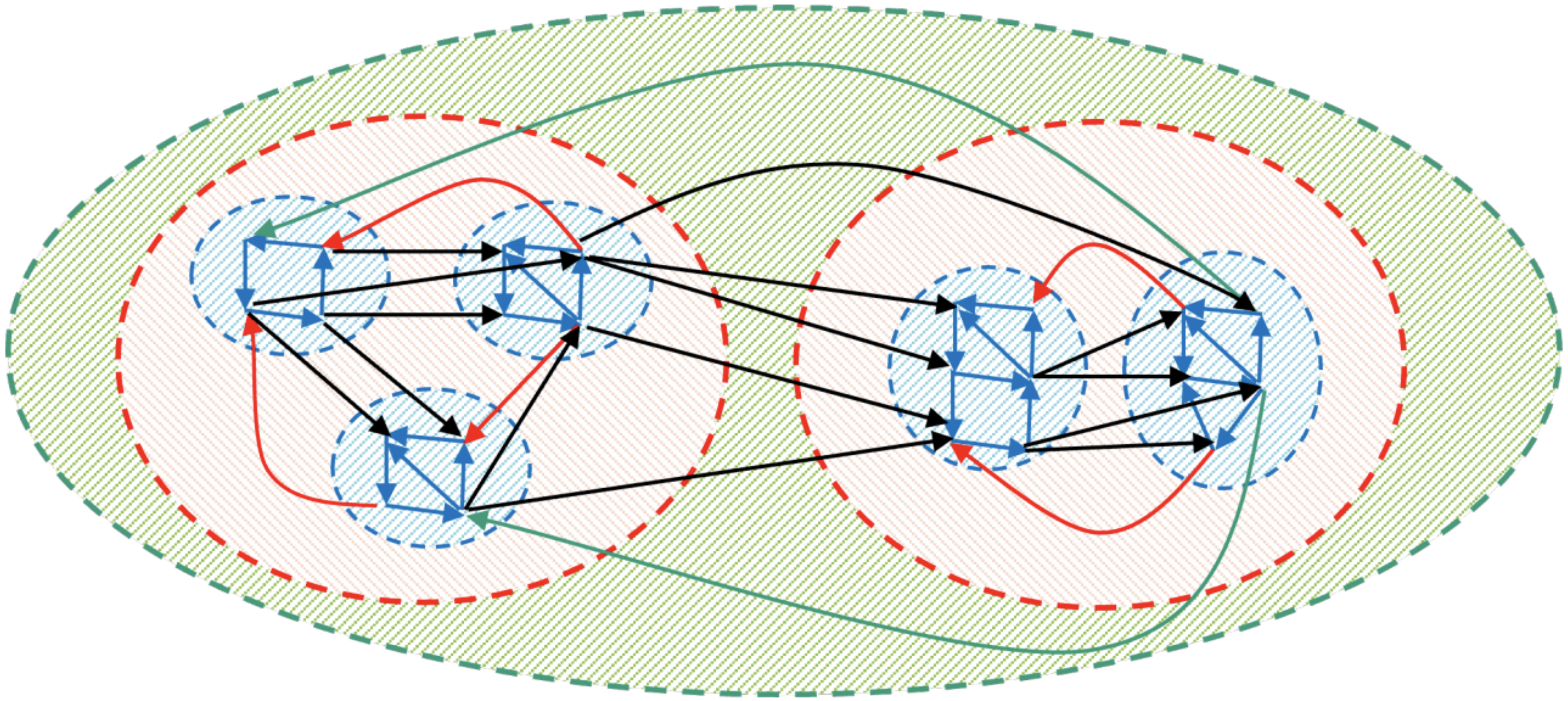

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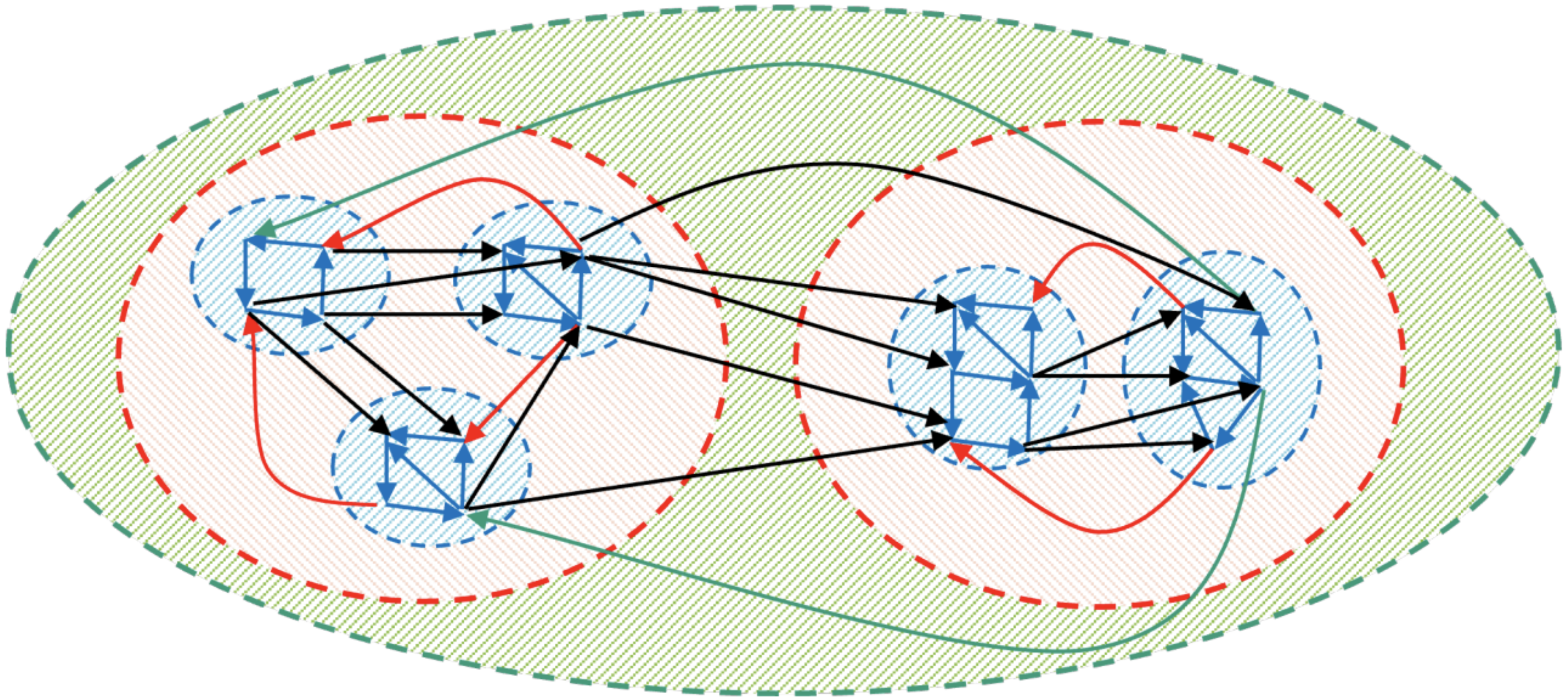

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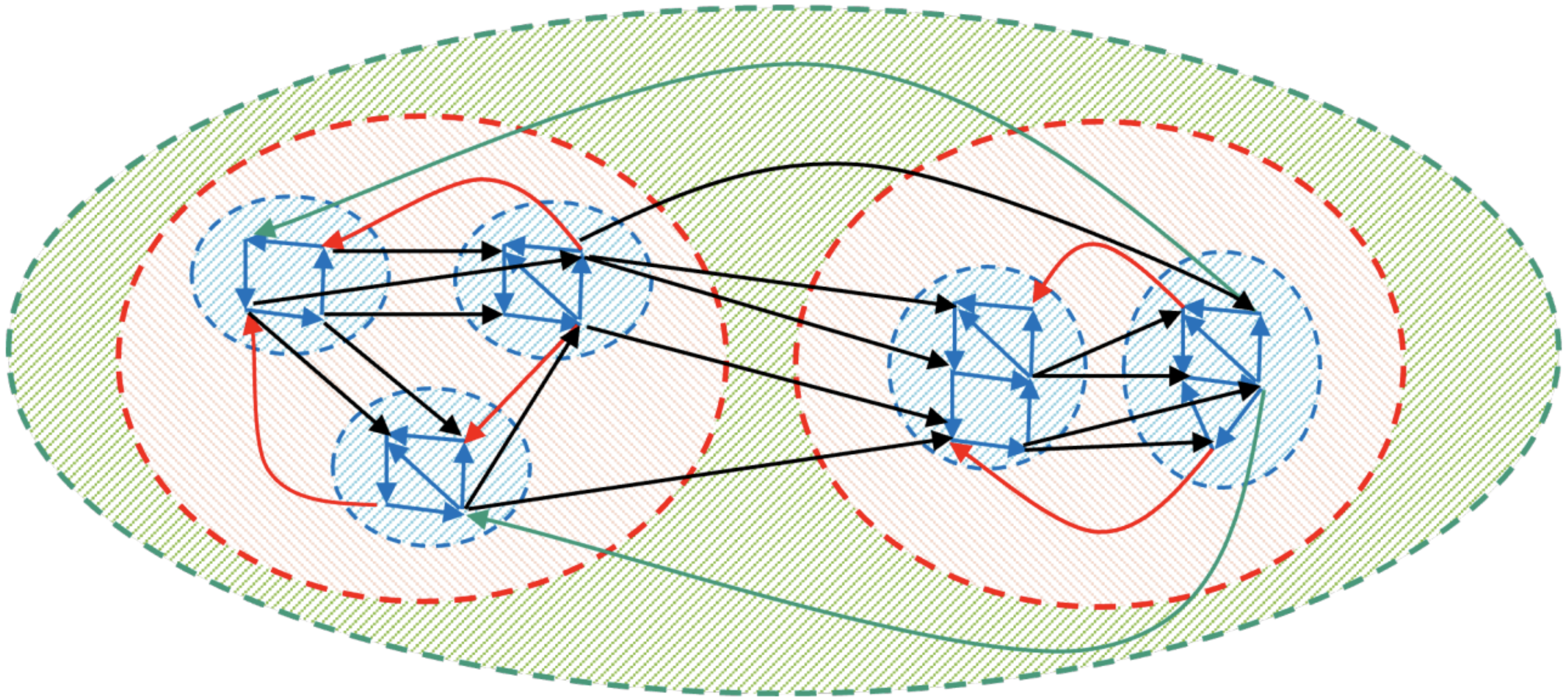

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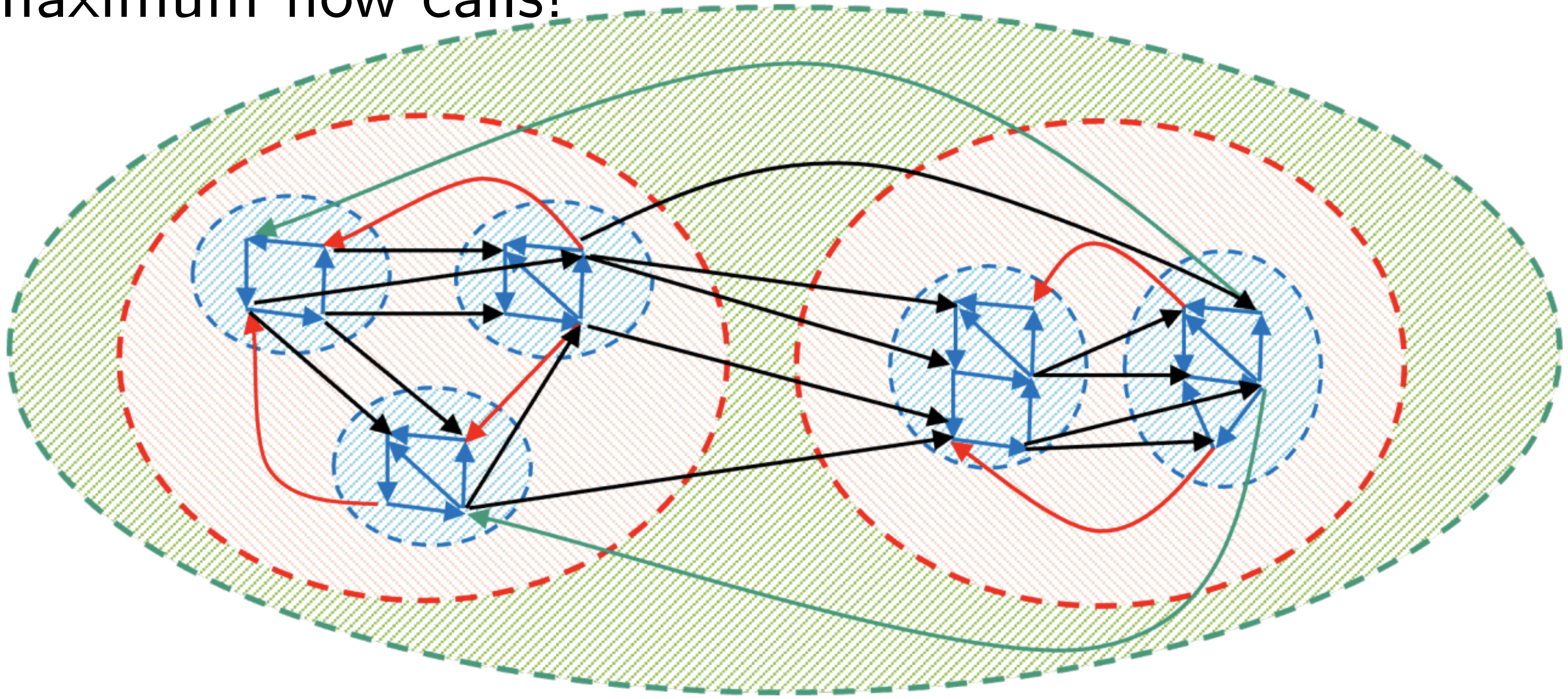

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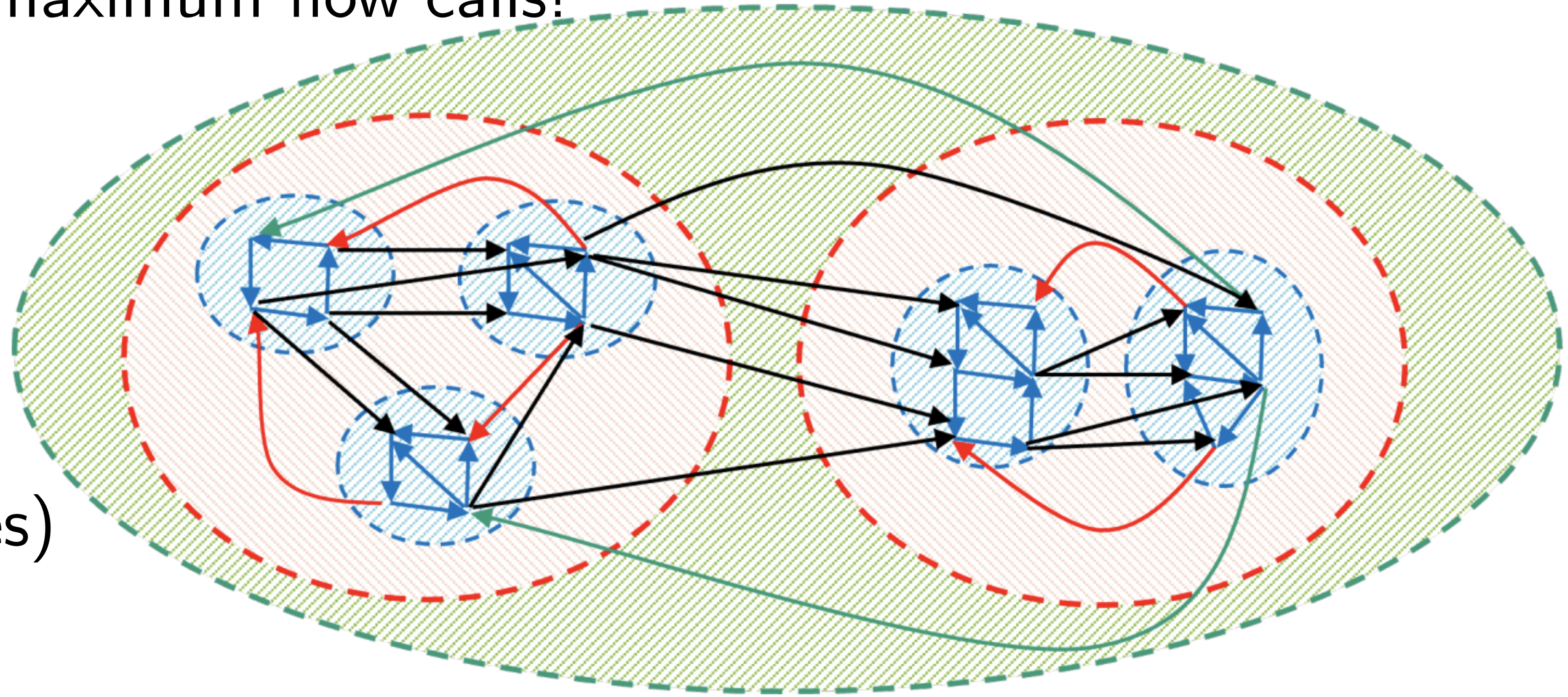
Build Bottom Up

Bootstrap Weighted P.R.

(solve "easier" flow instances)



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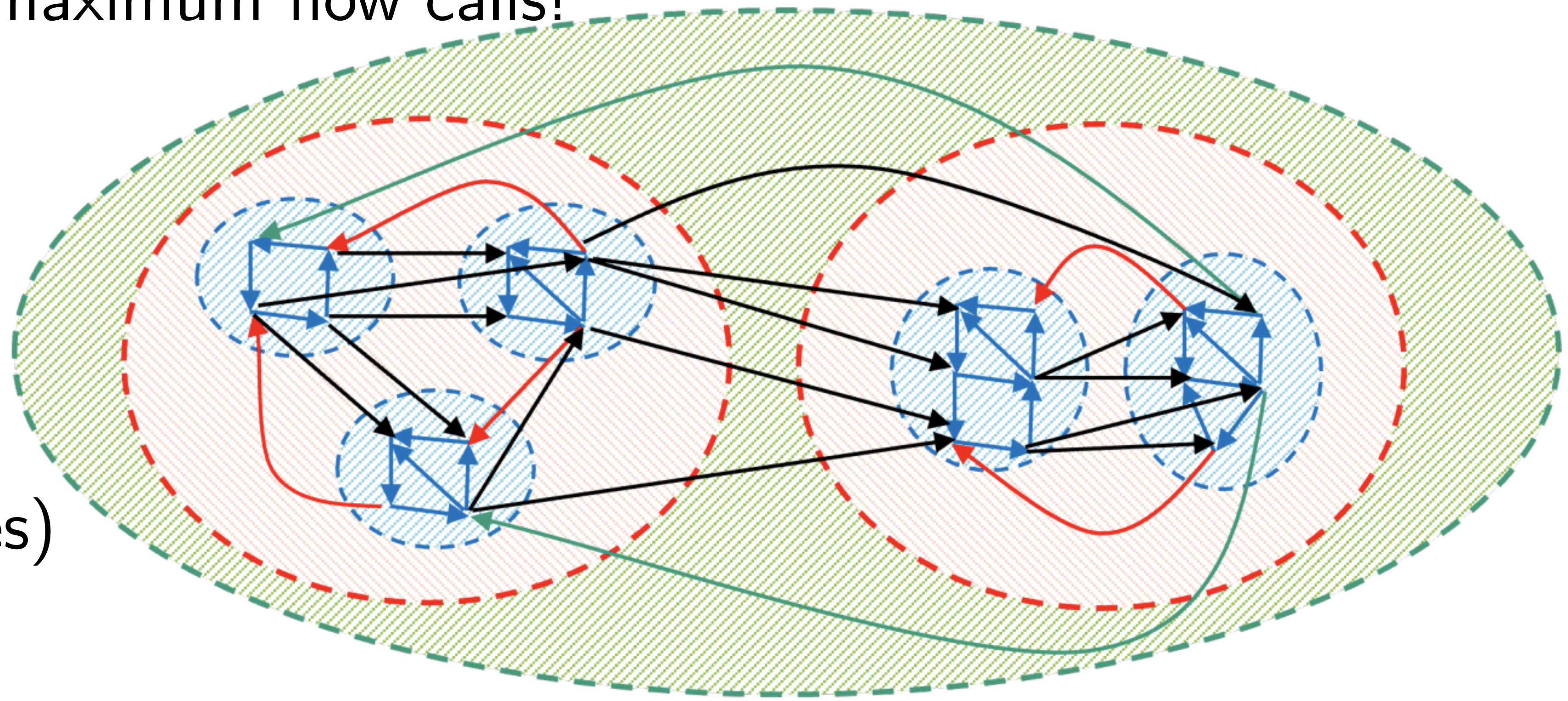

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Technically Complicated :(

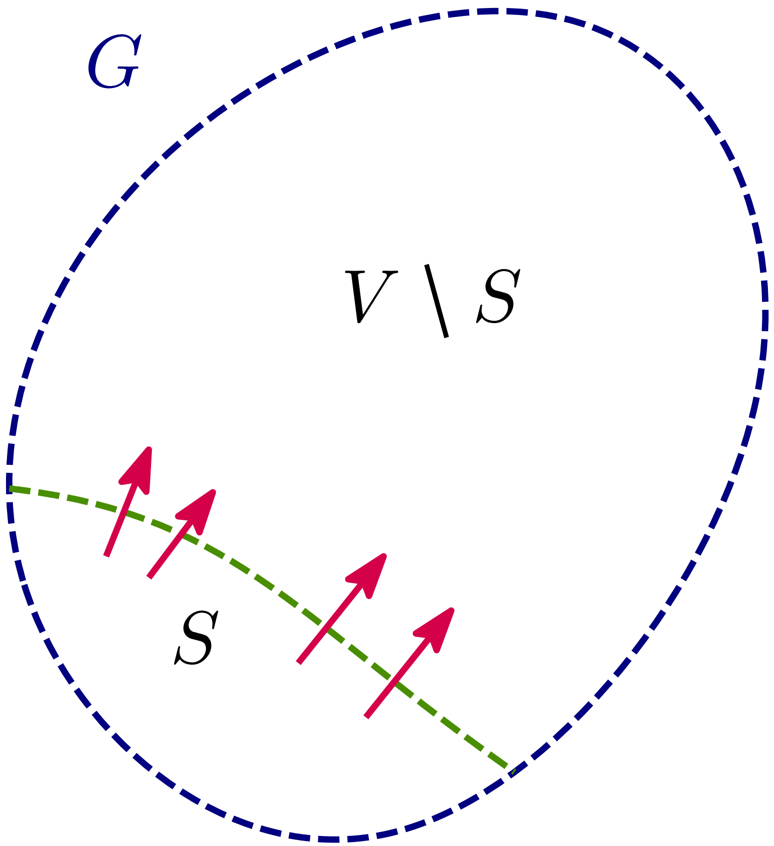
(bad guy: nestedness)

(half of our 99 page paper)

(Directed) Expanders

Def: G is ϕ-expander if $E(S, V \setminus S) \geq \phi \cdot \min\{\text{vol}(S), \text{vol}(V \setminus S)\} \quad \forall S$

$(\text{vol}(S) = \sum_{v \in S} \deg(v)$, $\phi \approx 1/n^{o(1)})$



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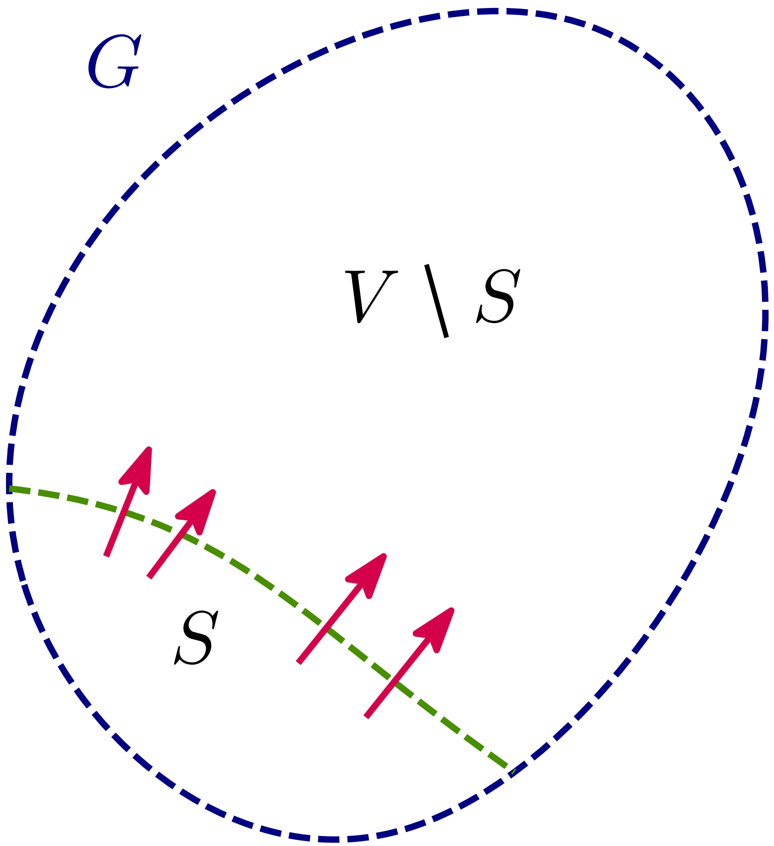
Examples:

Cliques

Bidirected Stars

Random



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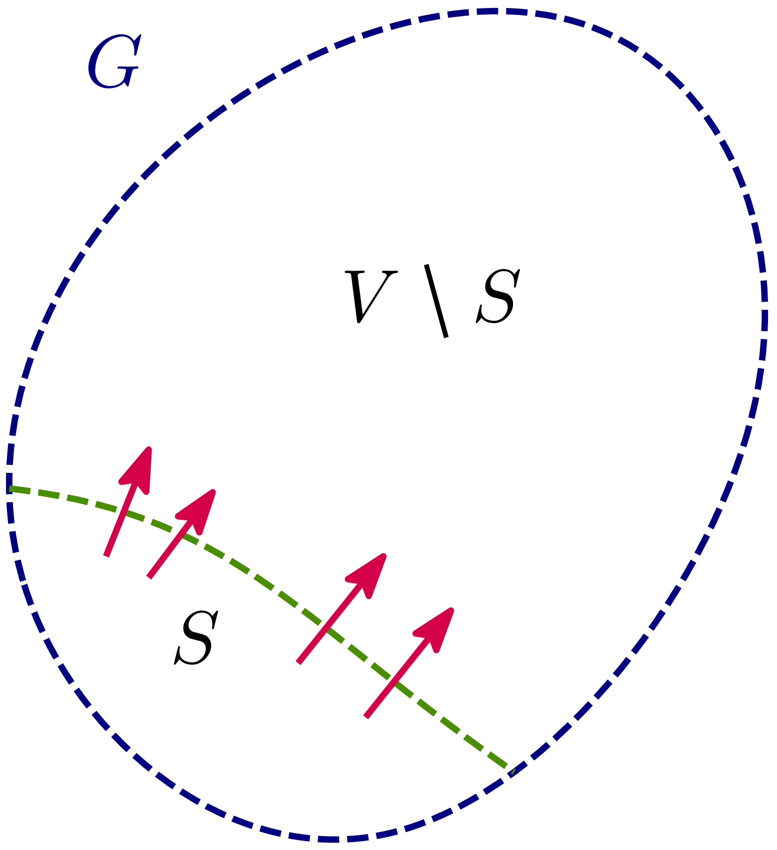

Examples:

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Why?

Well-connected

Low diameter $\frac{\log(n)}{\phi}$

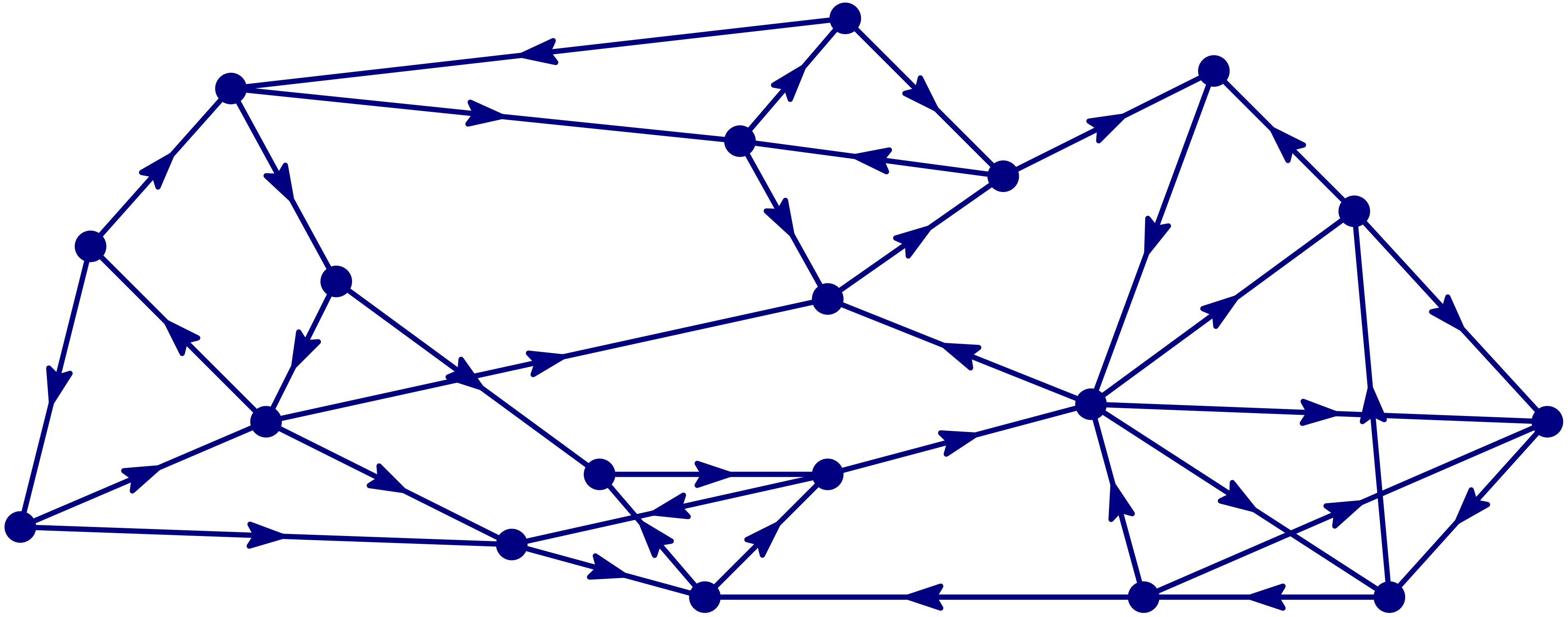
Easy to route (short) flow in

Robust to small changes

(Directed) Expander Decomposition

Every graph can be decomposed into:

1.
2.
3.



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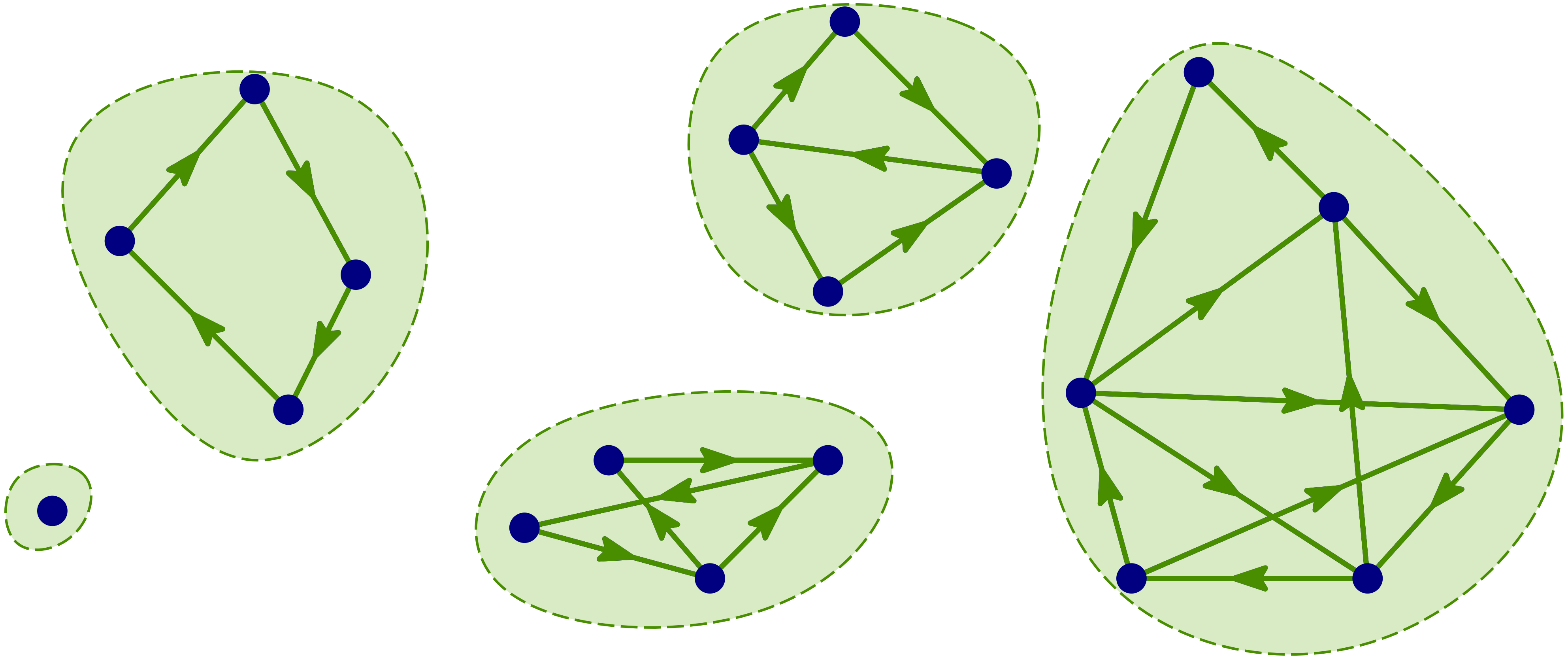

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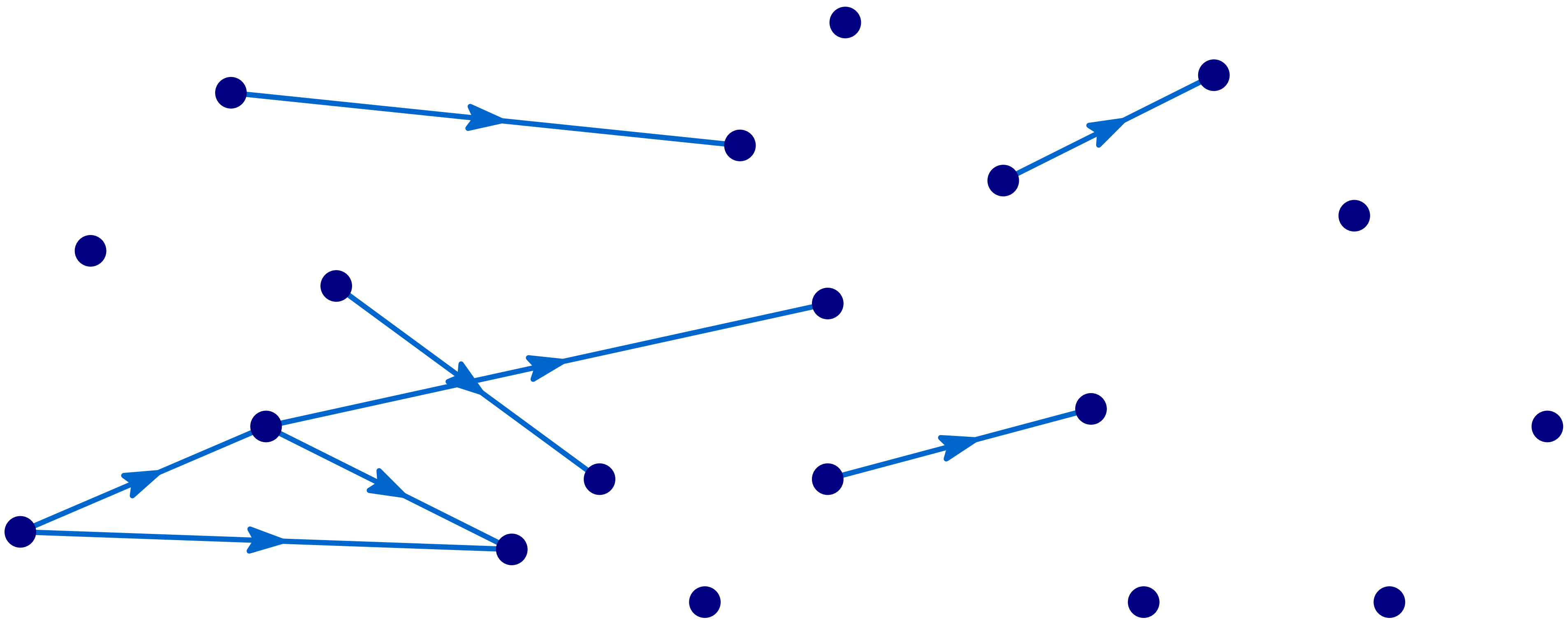

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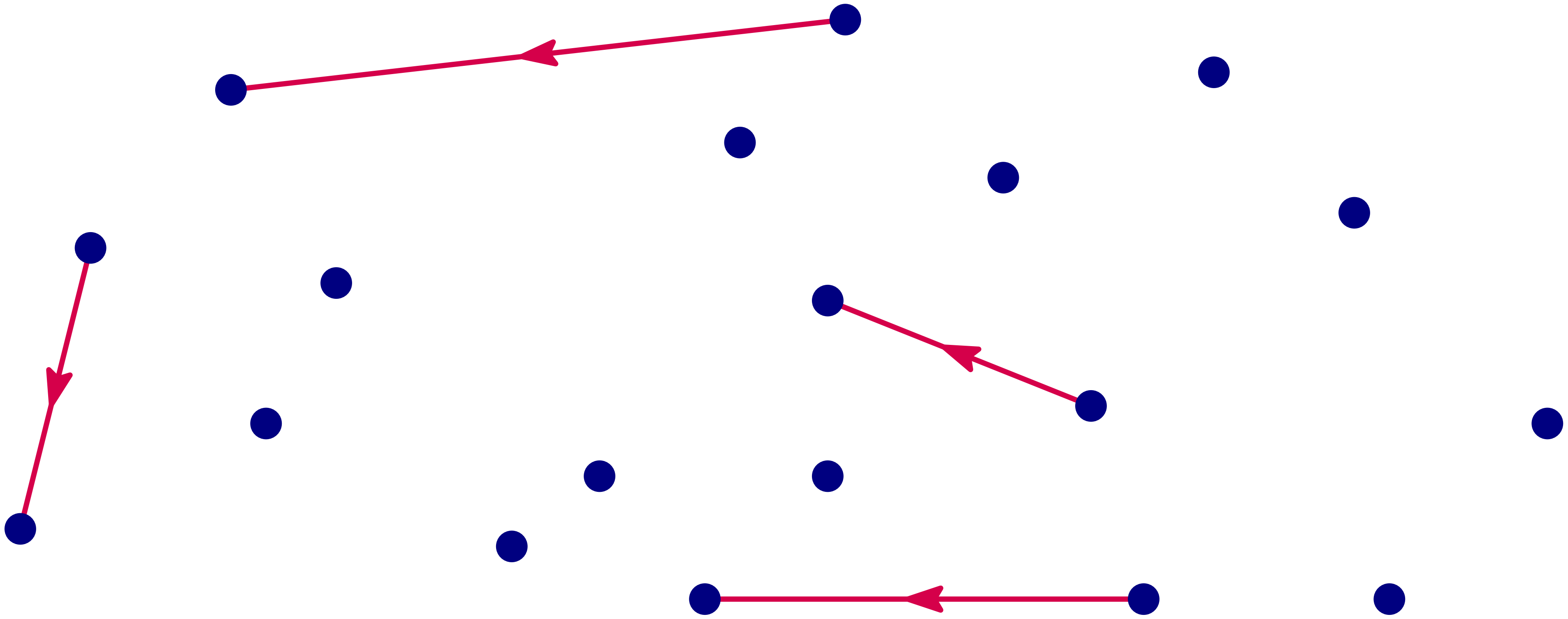

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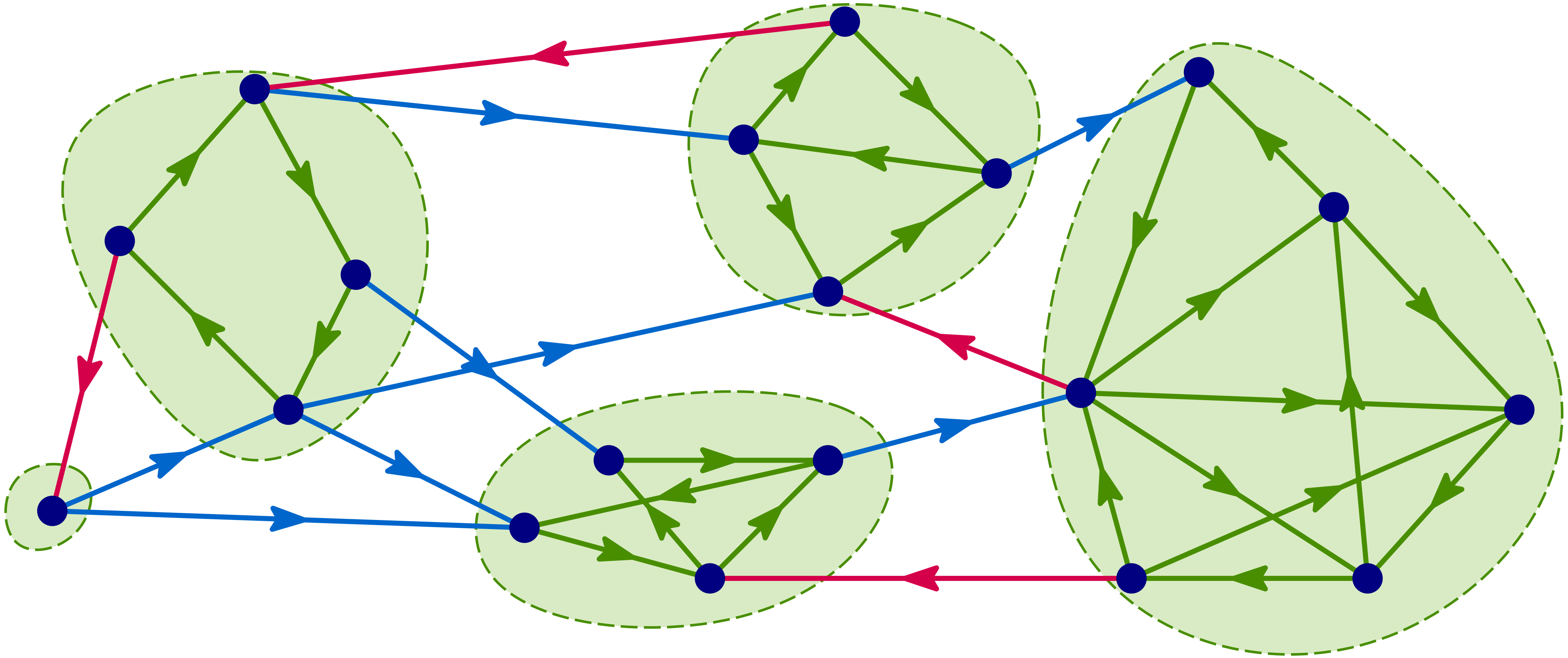

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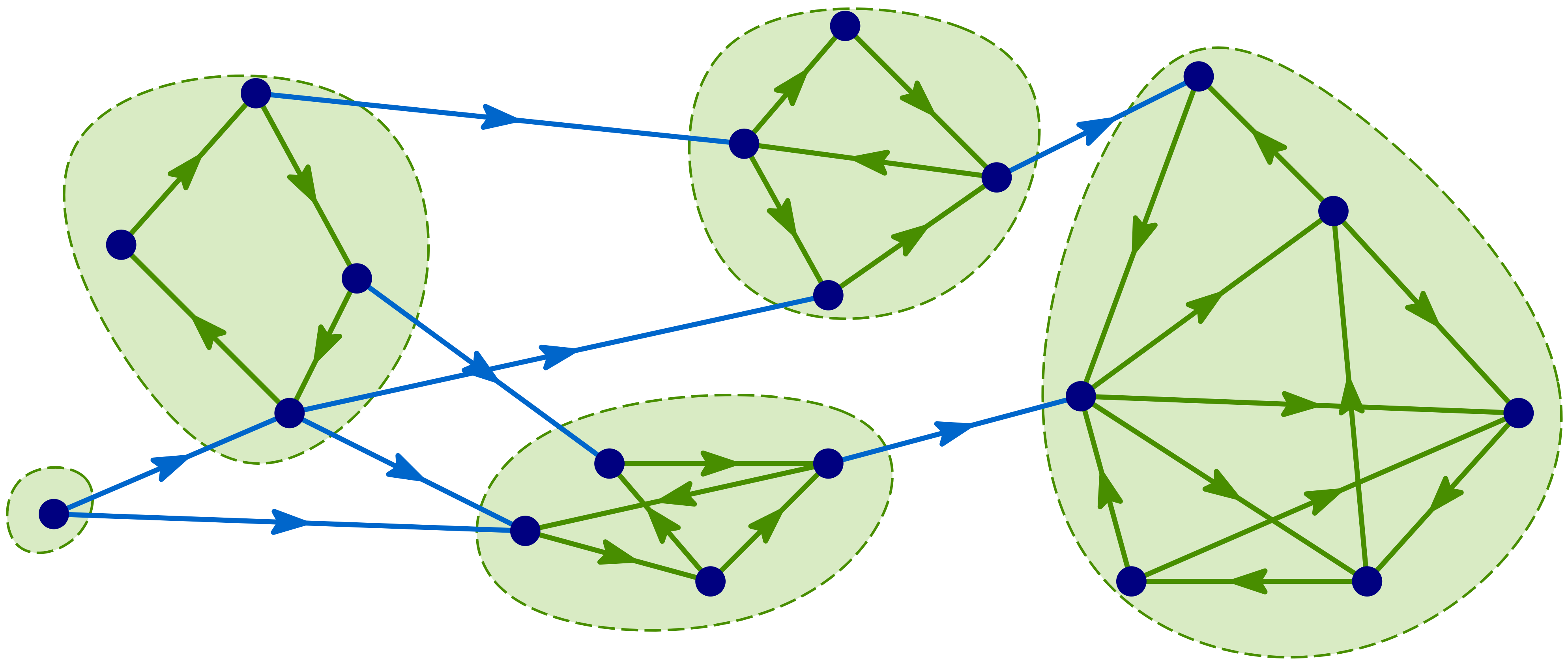

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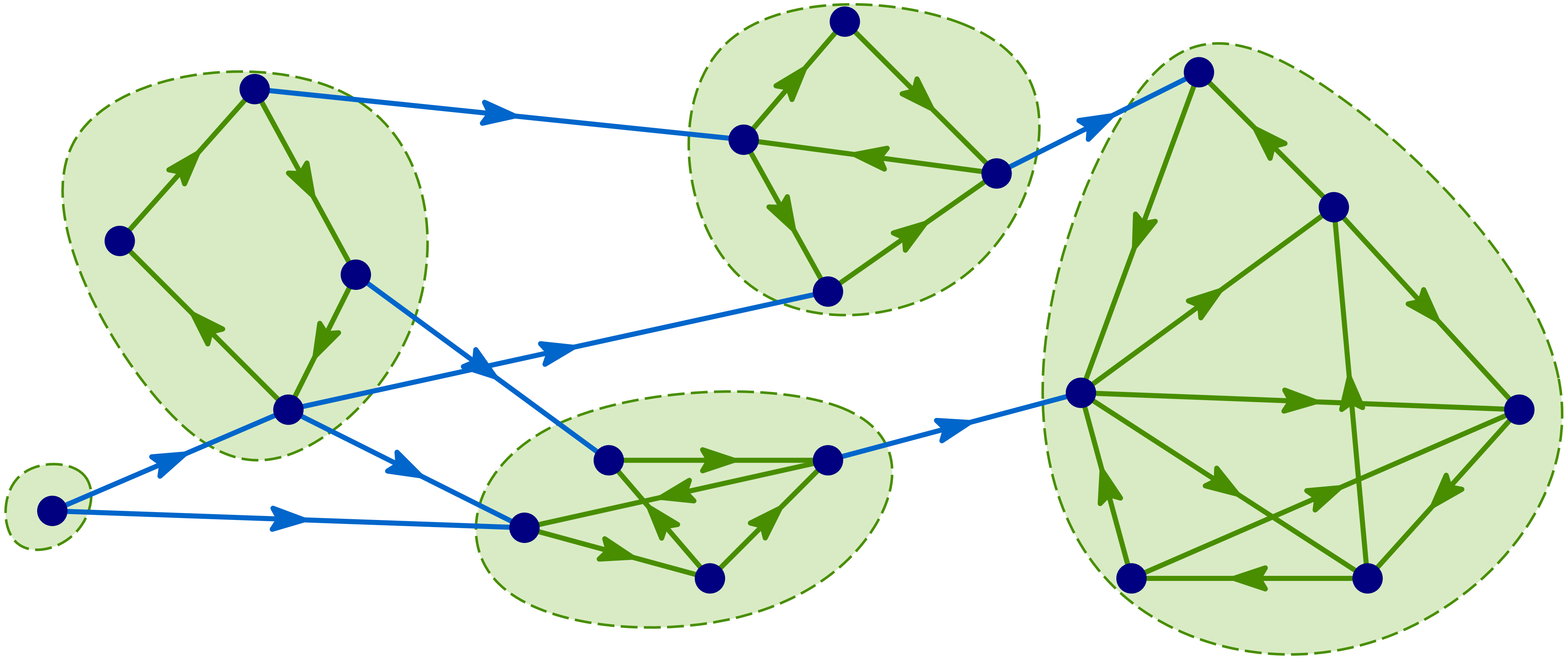
$\boldsymbol{\tau}$ respects DAG

$\boldsymbol{\tau}$ contiguous in expanders



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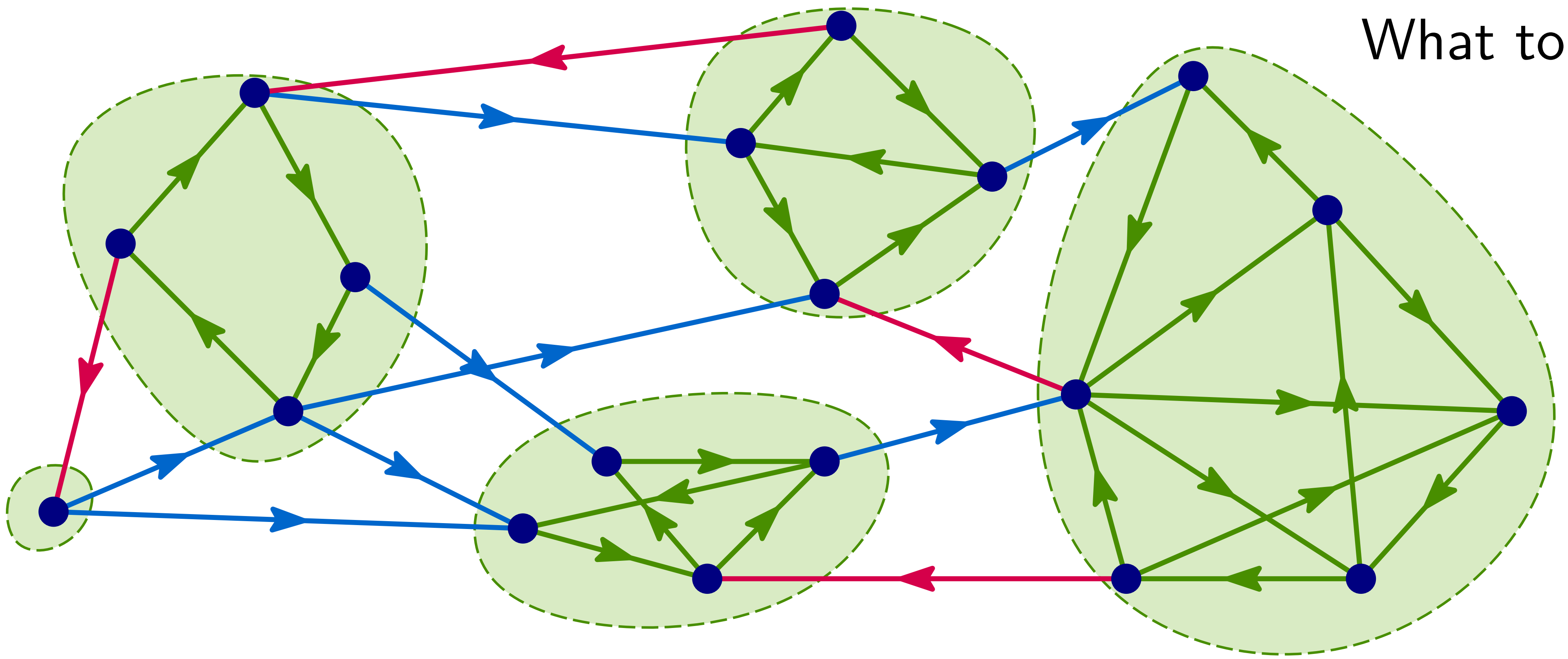

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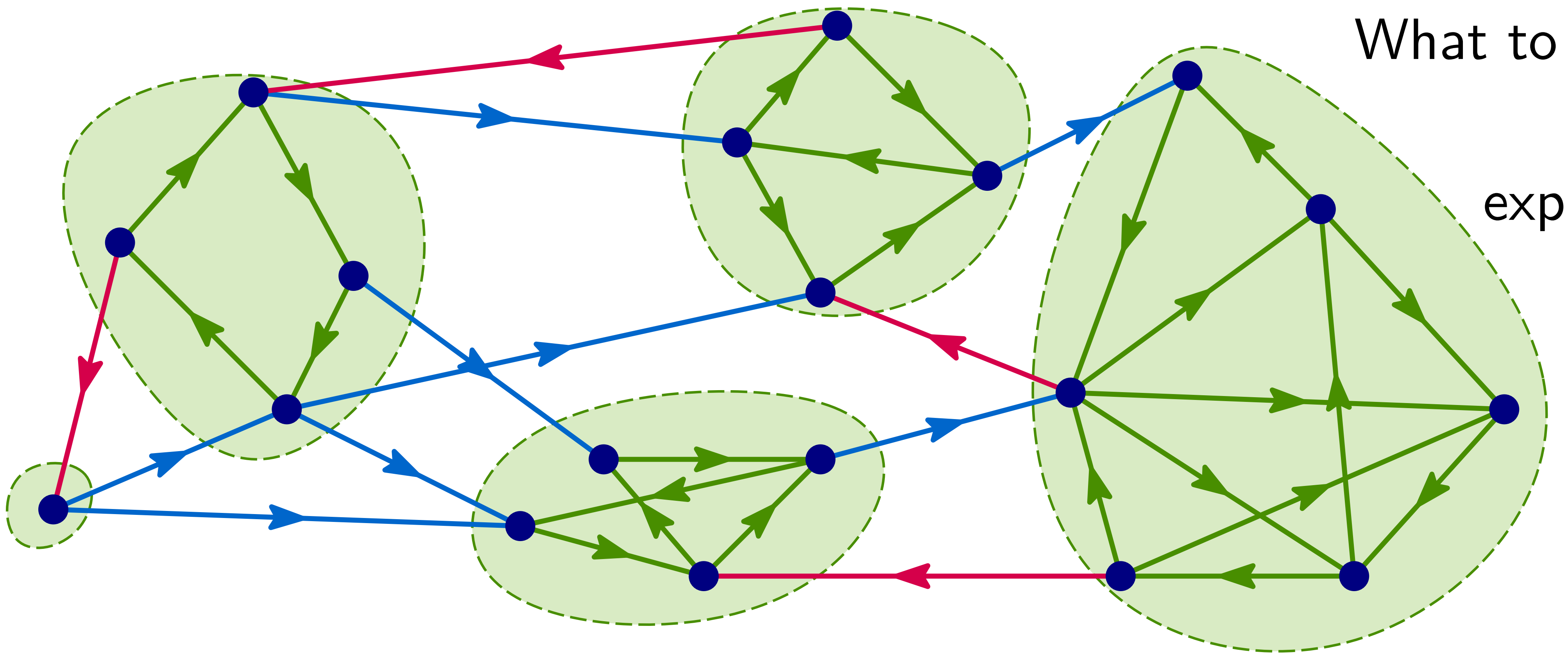


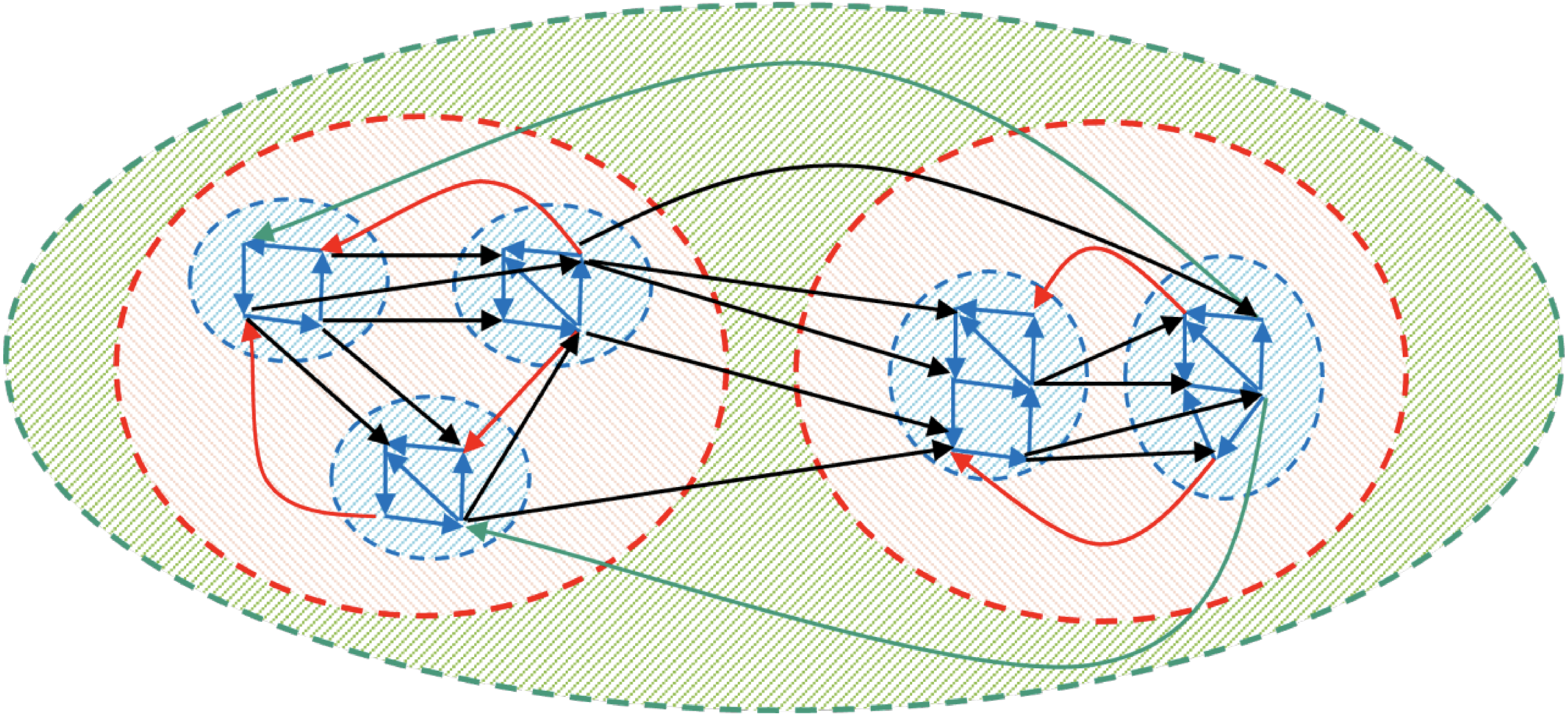
What to do about B?

recurse!

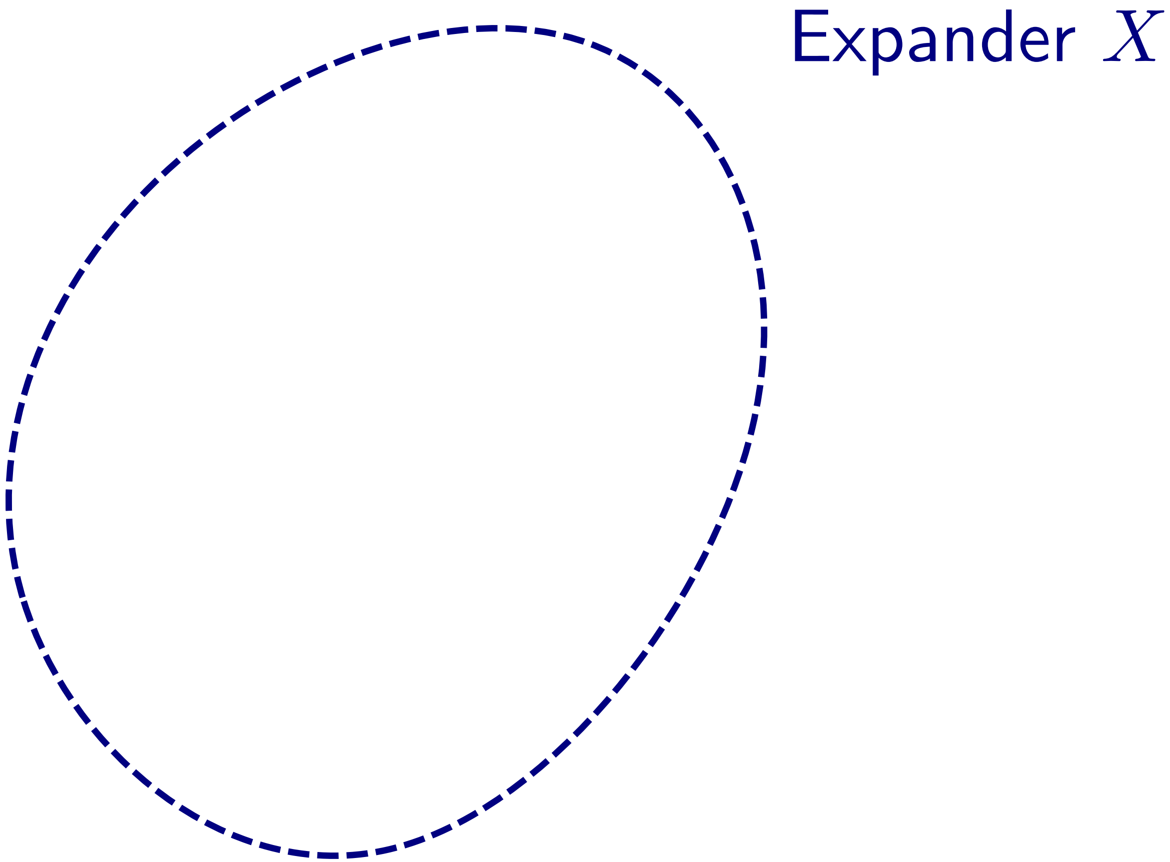
expander decomp.

w.r.t B

(Directed) Expander Decomposition



Technique Highlight: Path-Reversal Expander Pruning



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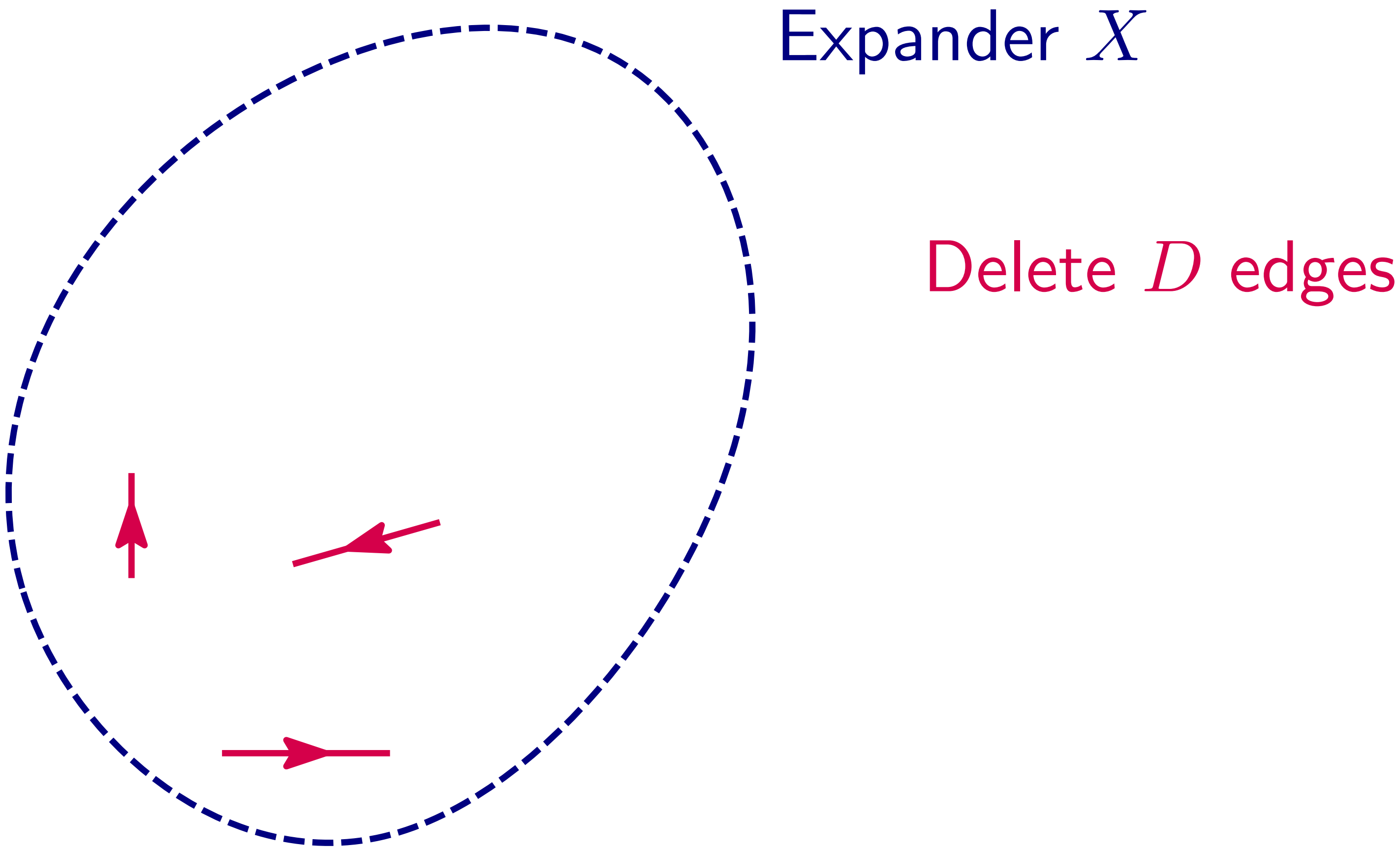

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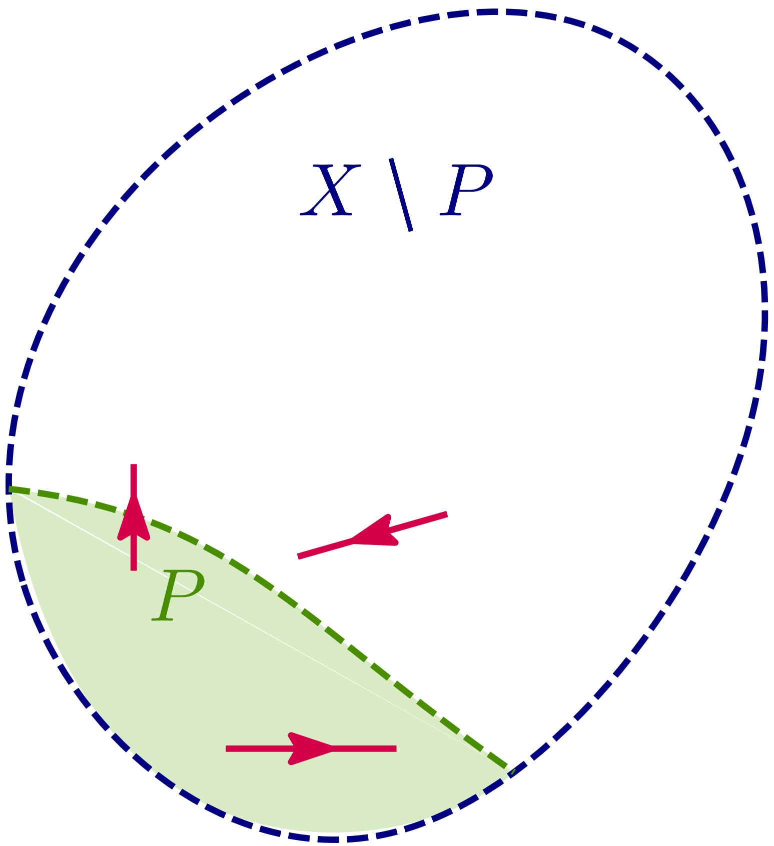


Expander X

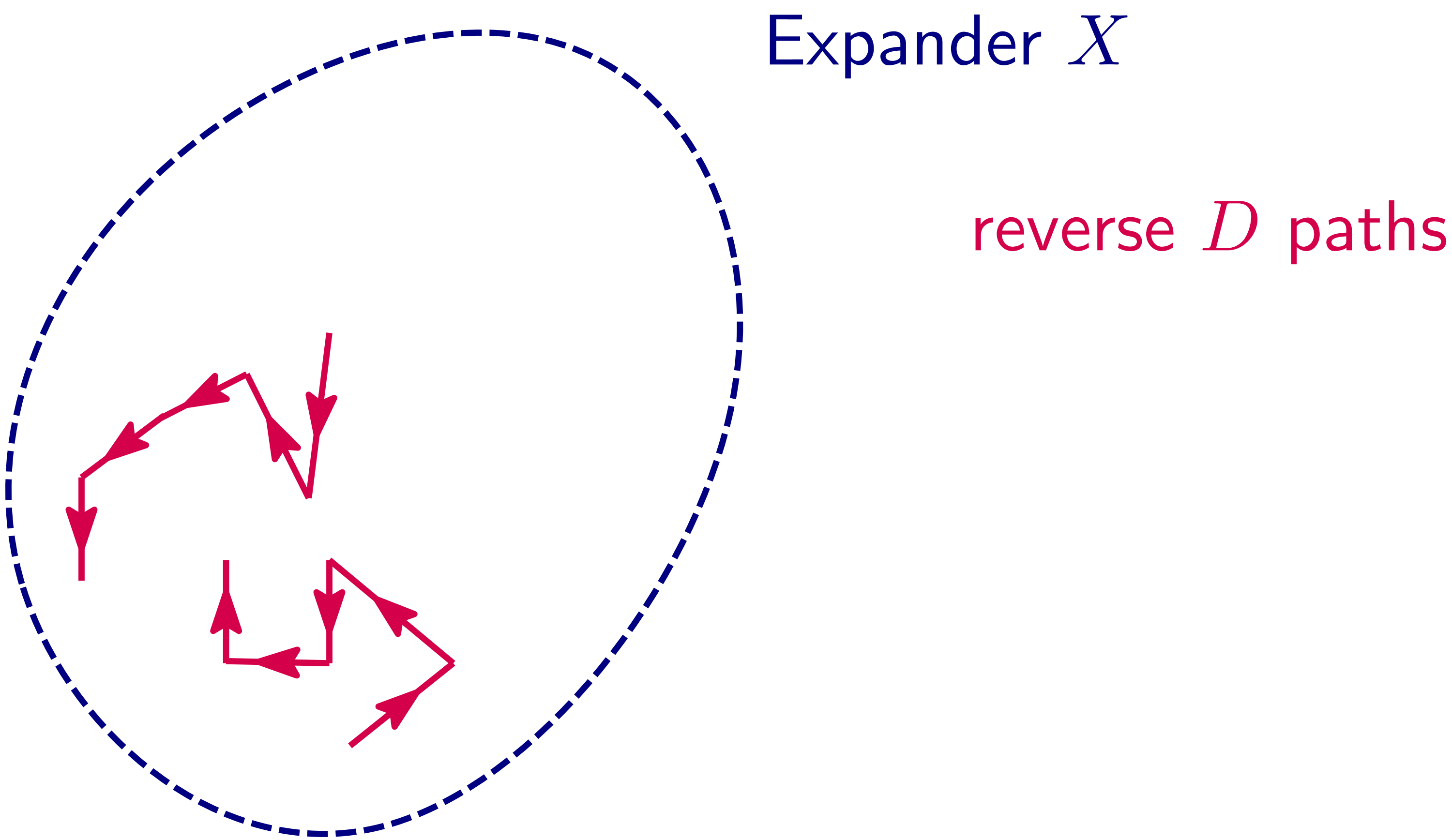
Delete D edges

Small “pruned” part P $\mathrm{vol}(P) \leq 6|D|/\phi$

$X \setminus P$ is still expander

Known: “Expander Pruning”

Technique Highlight: Path-Reversal Expander Pruning



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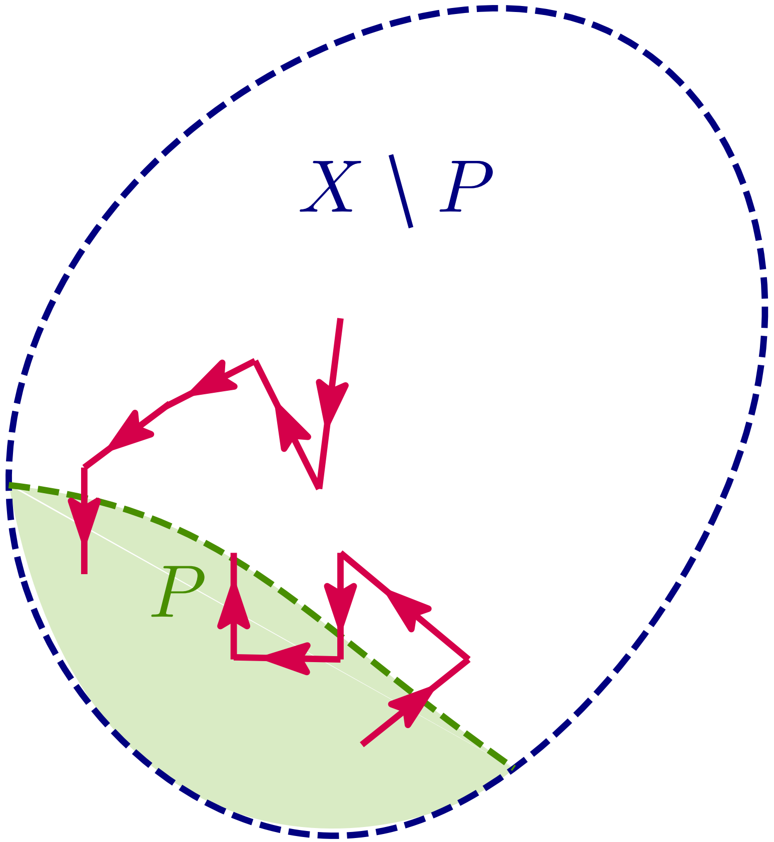

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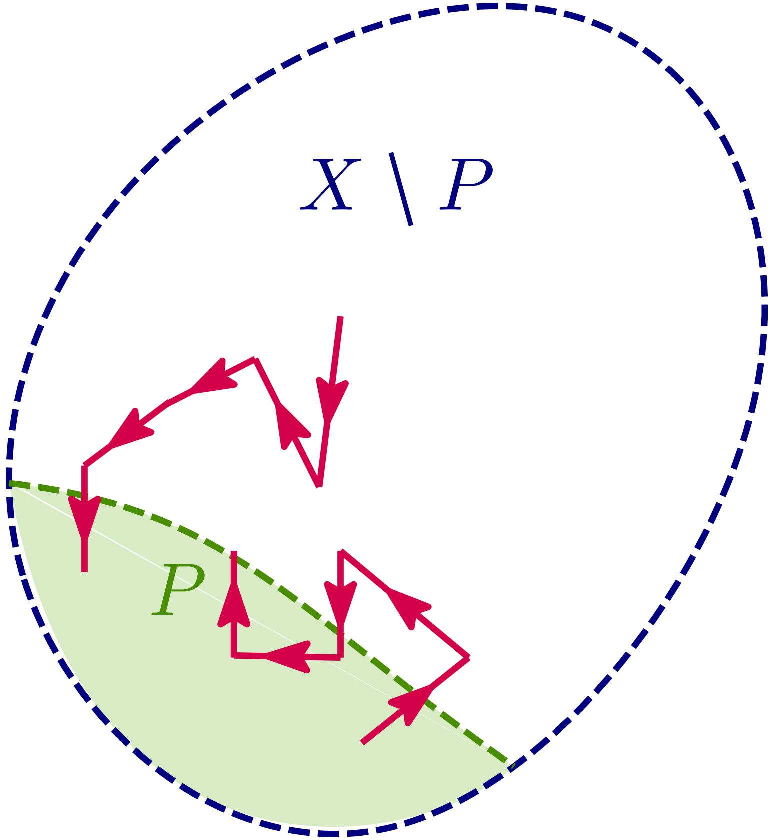

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Directed Expander Hierarchy is robust under flow augmentation

Bottleneck towards $\tilde{O}(m)$:
Approximate Max Flow in DAGs

Summary & Open Problems

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Ours

[Chen-Kyng-Liu-Peng-Gutenberg-Sachdeva'22]

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Maximum Flow	Minimum Cost Maximum Flow

Comparision

Ours

Maximum Flow

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[Chen-Kyng-Liu-Peng-Gutenberg-Sachdeva'22]

Minimum Cost Maximum Flow

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Combinatorial Augmenting Paths	Continuous Optimization Dynamic Data Structures
Implementable?	Tricky to implement