Minimum Star Partitions of Simple Polygons in Polynomial Time

Mikkel Abrahamsen[1]
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André Nusser[1,3]
Hanwen Zhang[1]

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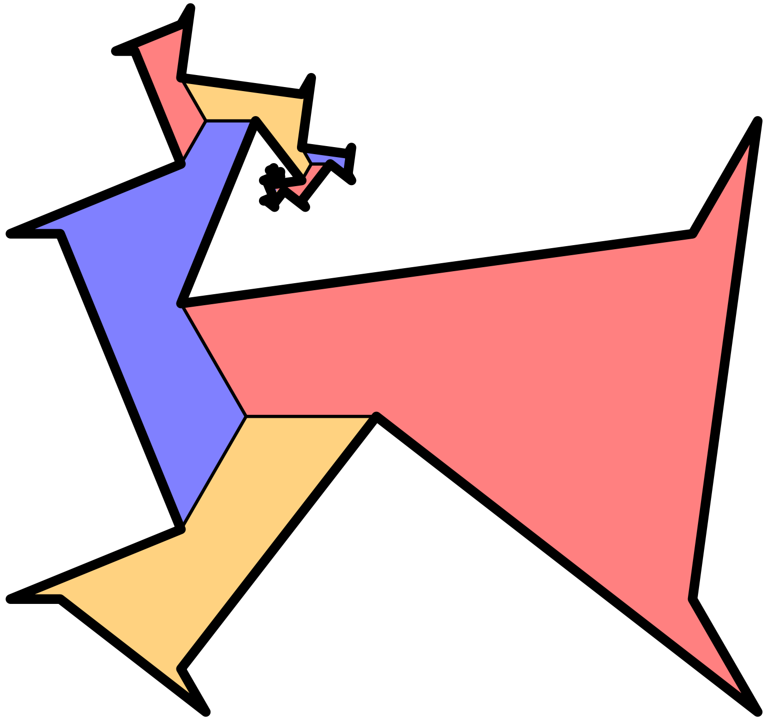

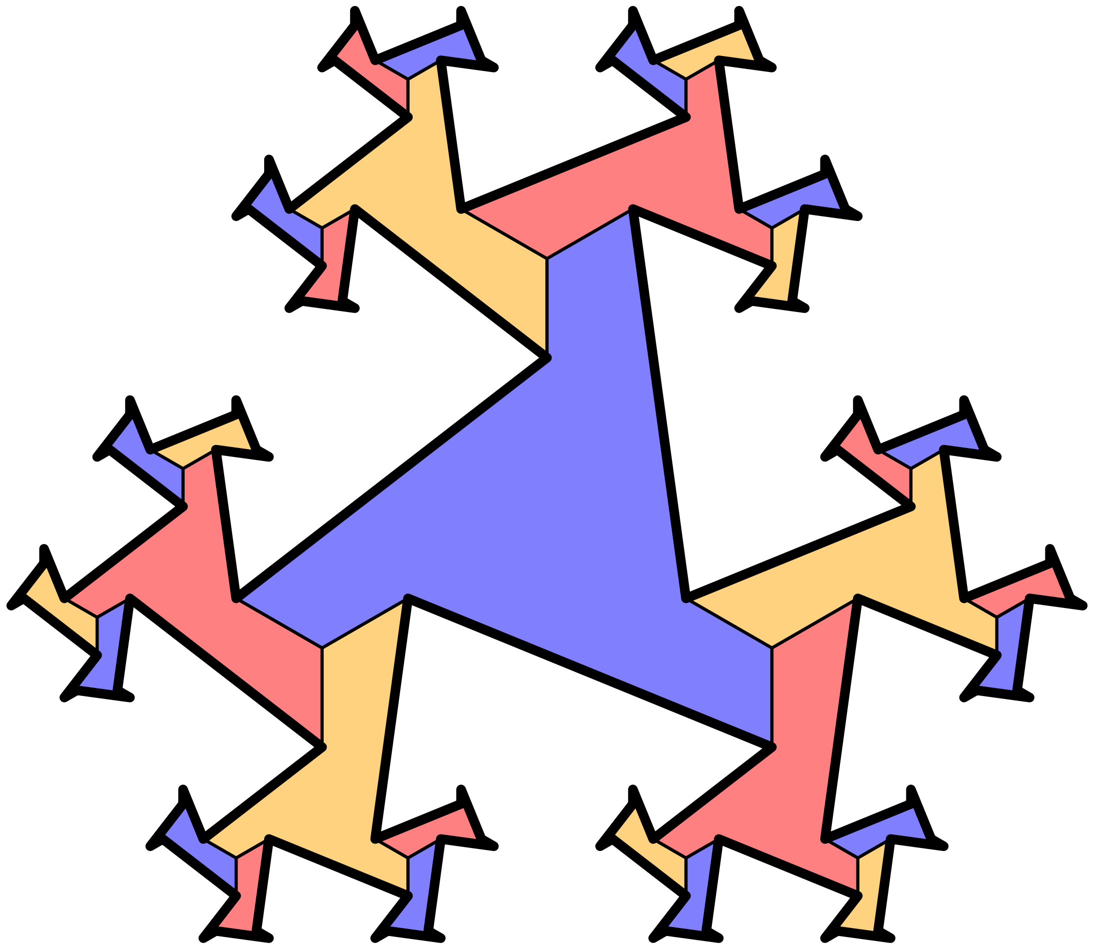

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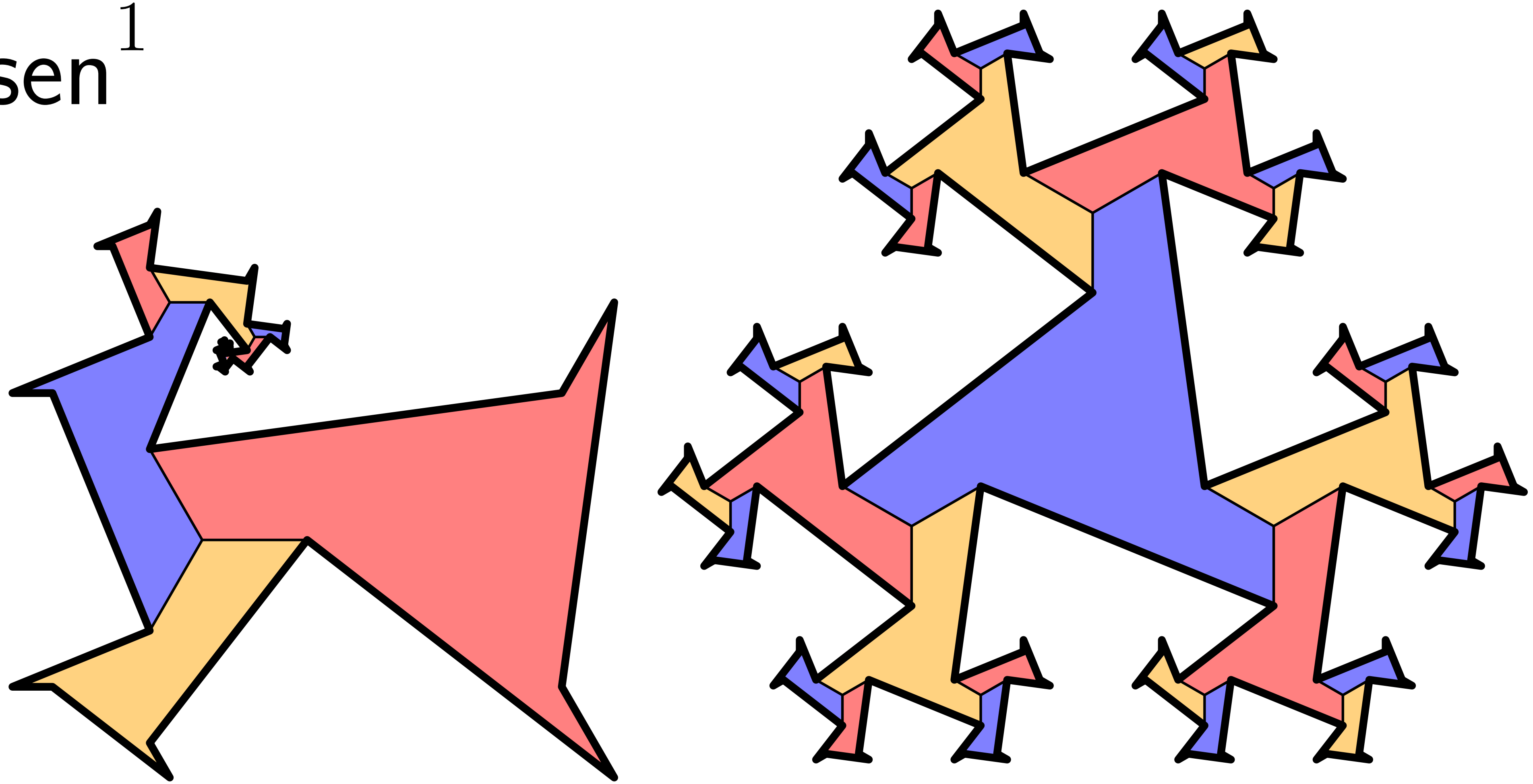
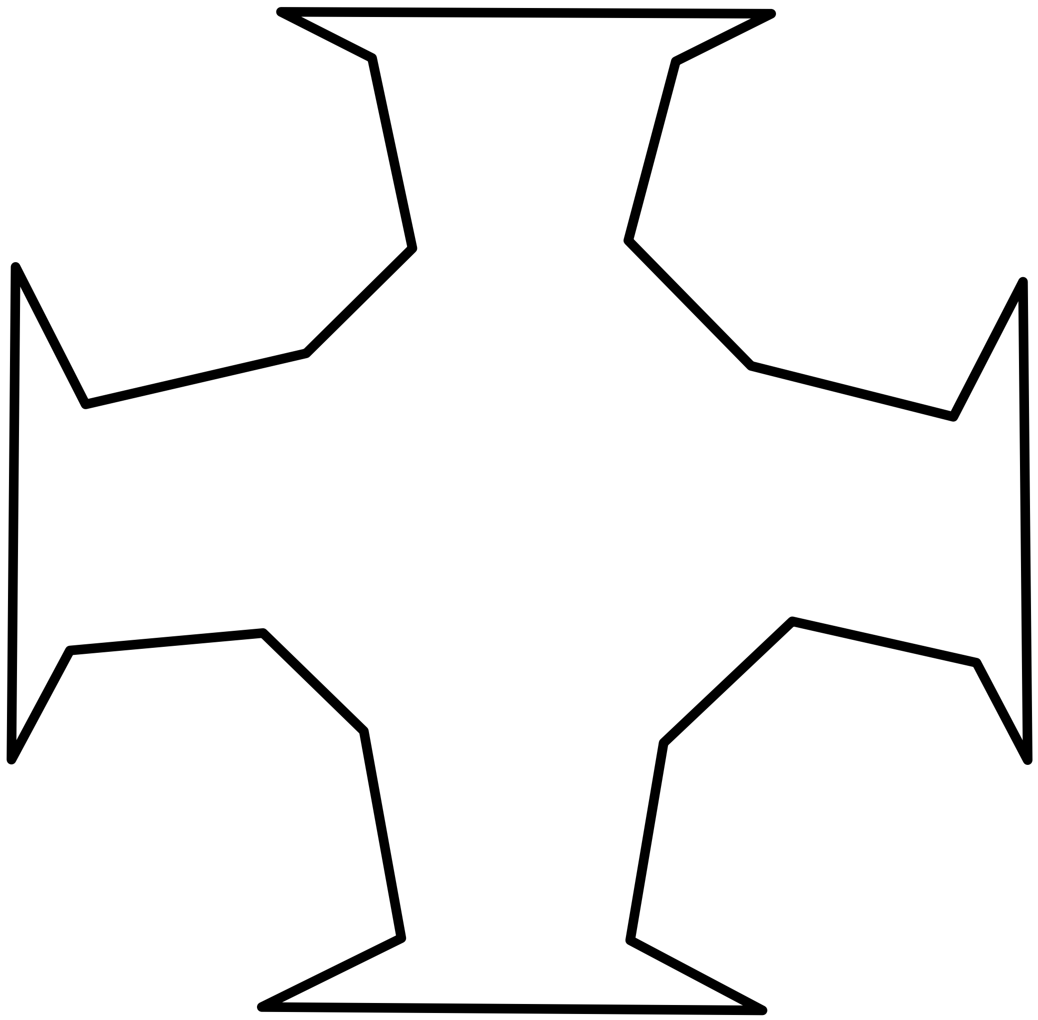


Suprising Open Problem — Triangle Partitions

Given polygon (n = #corners),
Partition into few triangle pieces



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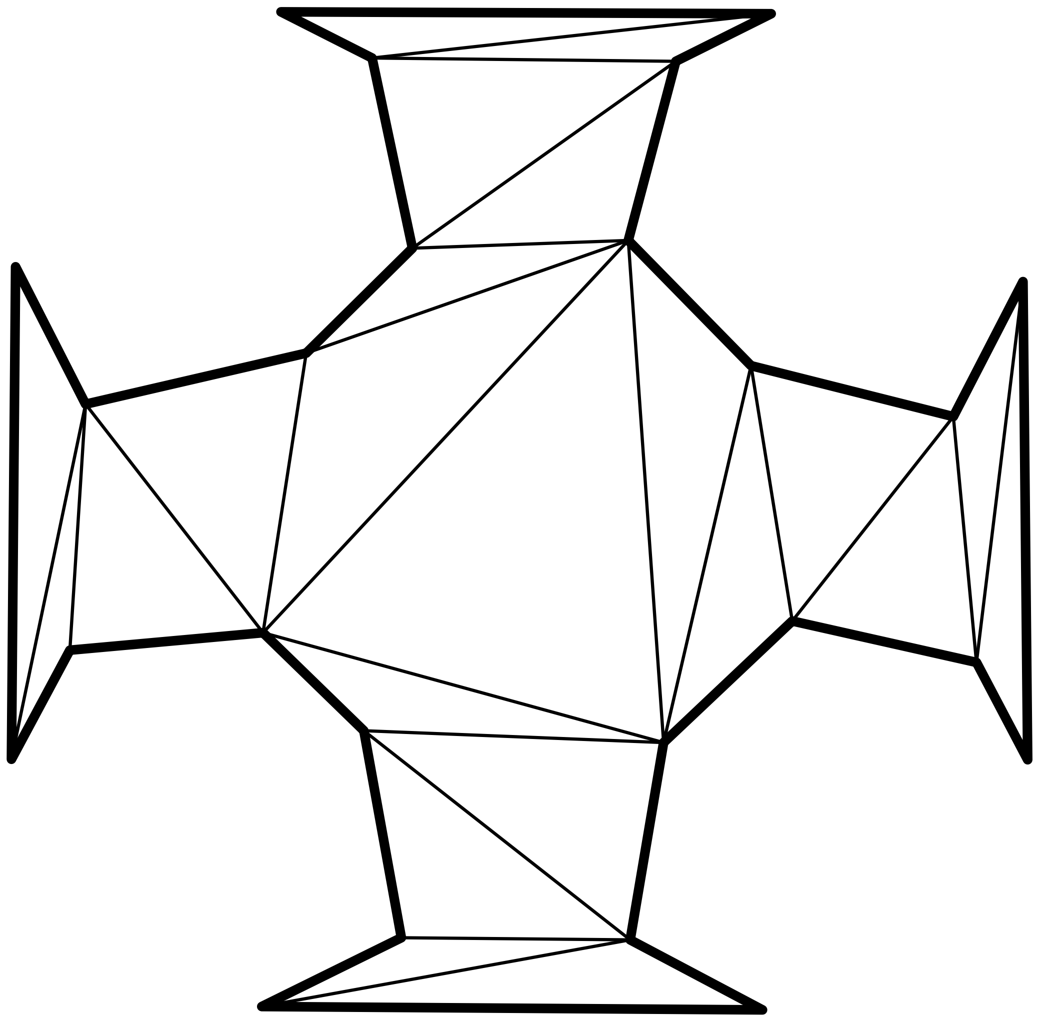
Triangulation: $n - 2$ triangles **22**



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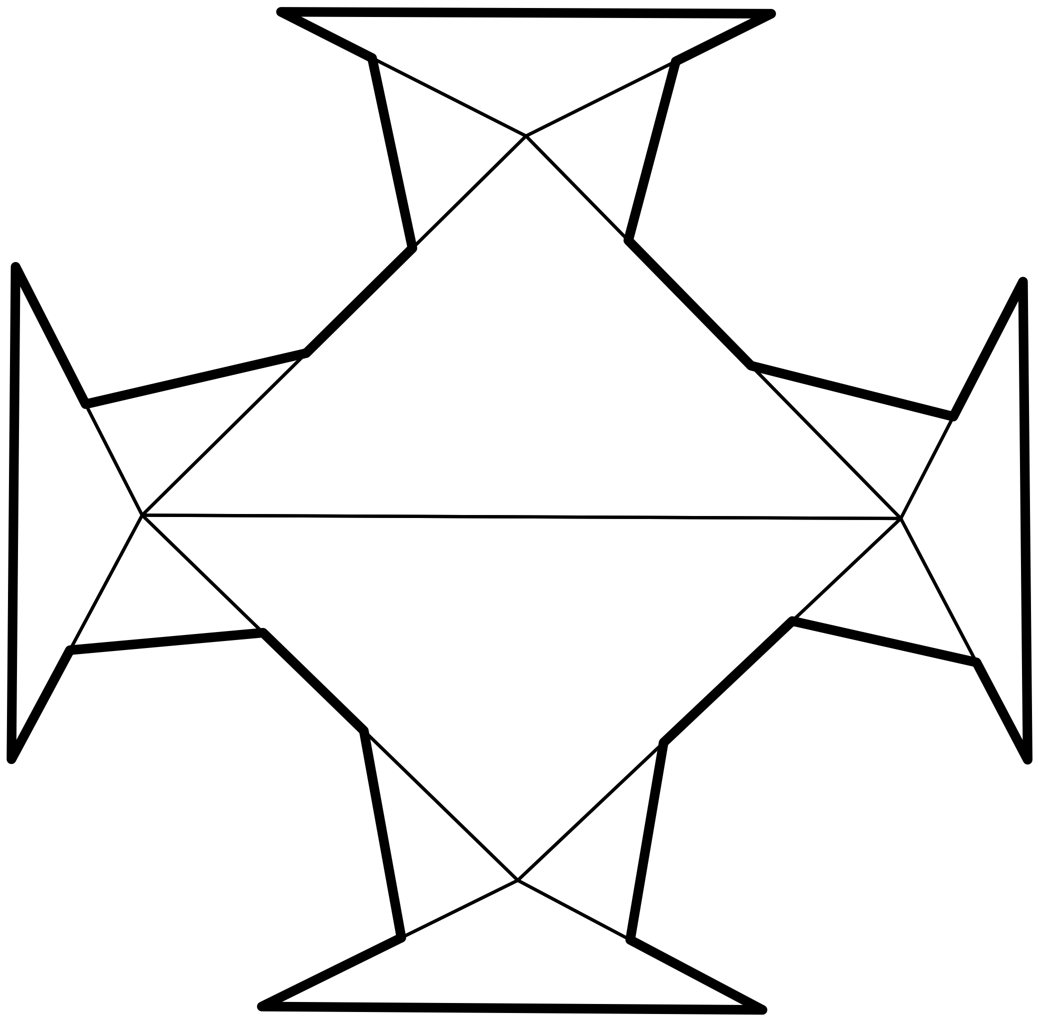
Not always optimal! **14**



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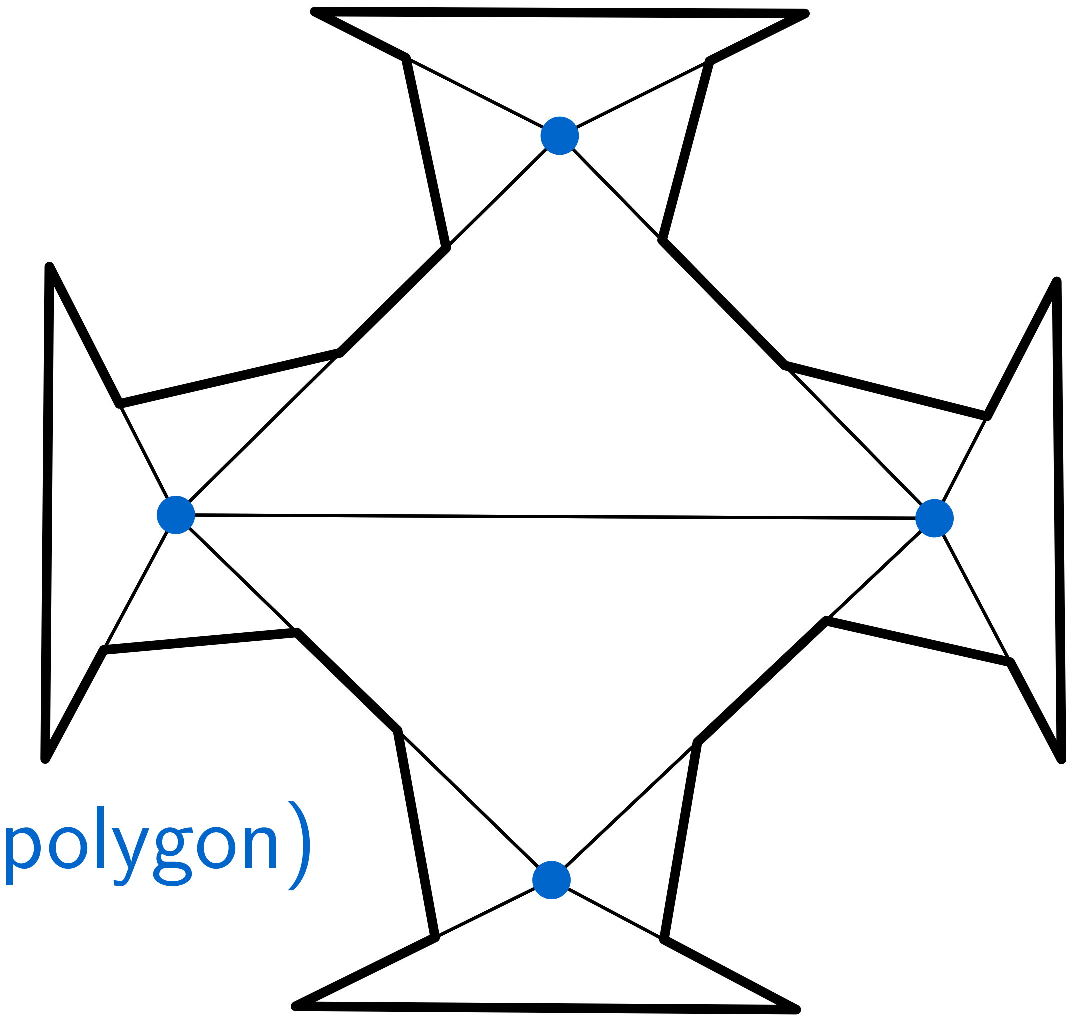

Difficulty: Steiner Points

(corner of the solution but not of input polygon)

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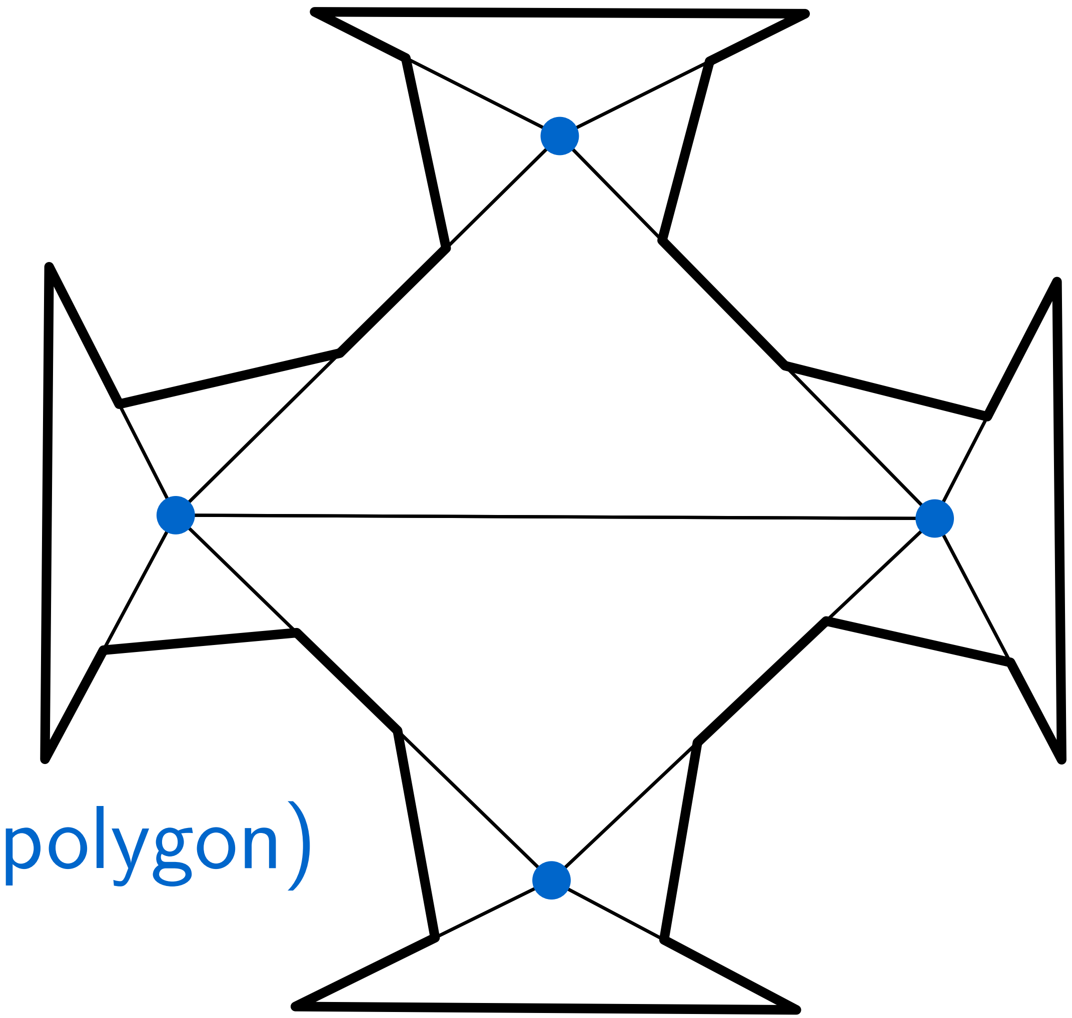

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Open Problem: Polynomial time?

Star Partitions

This Talk: Triangle Partitions

Star Partitions

This Talk: **Star** Partitions

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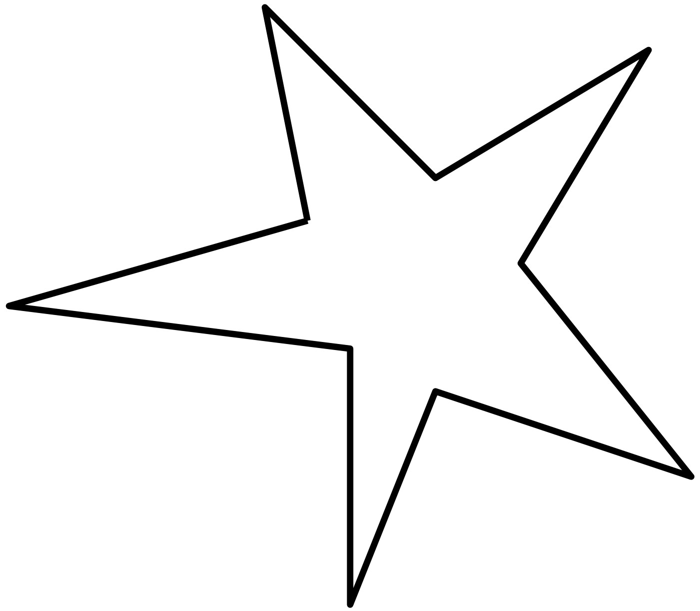
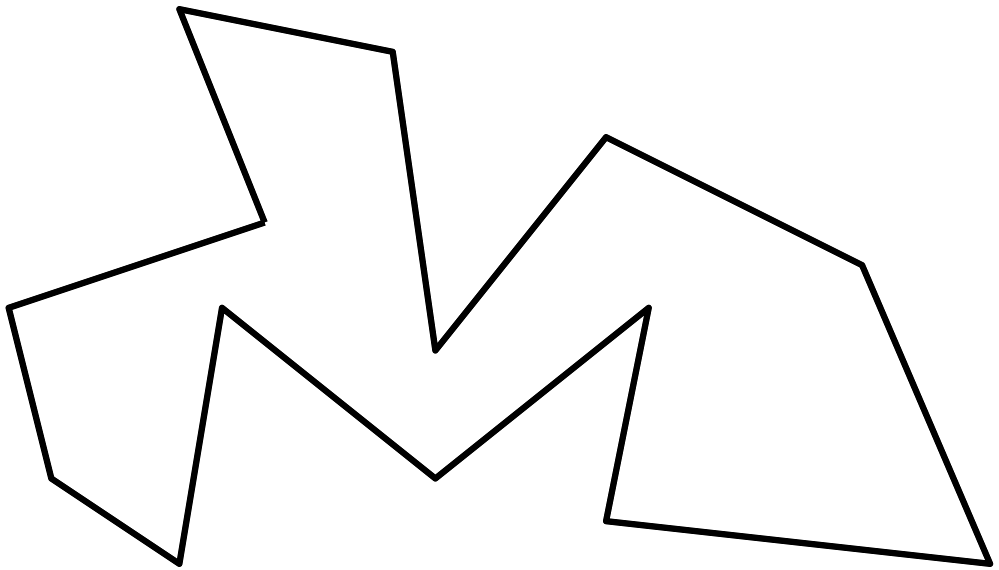


Star

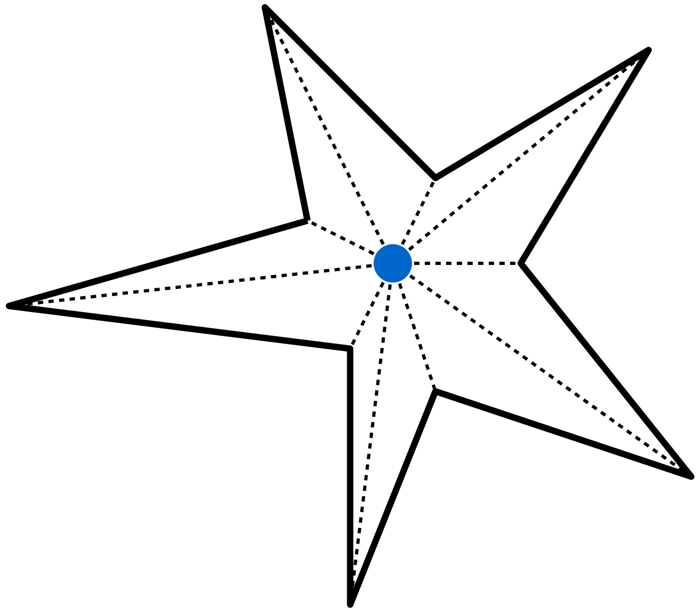


Not a Star

Star Partitions

This Talk: Star Partitions

Definition: Star iff exists star-center point which can see all of the polygon



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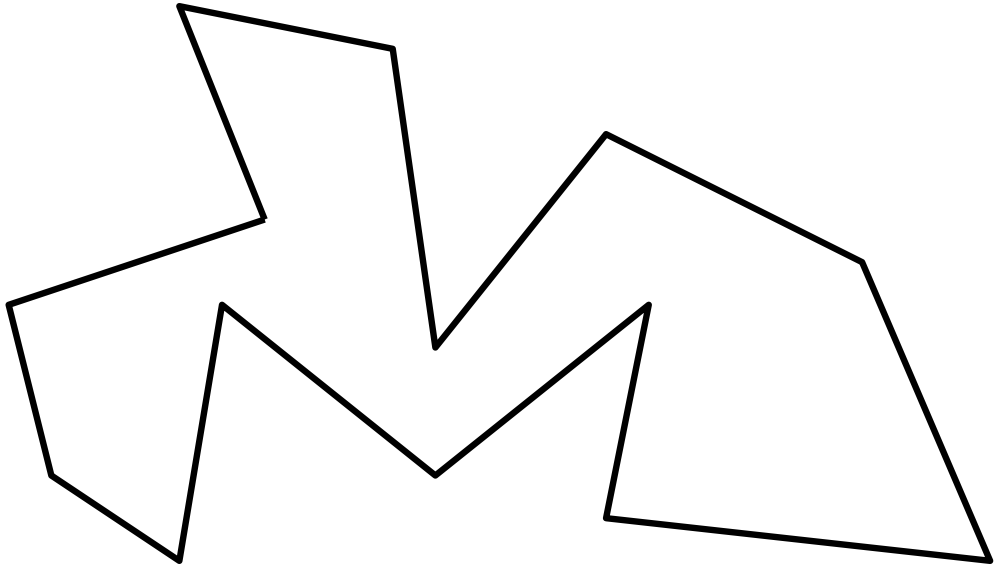

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Star Partitions of Polygons

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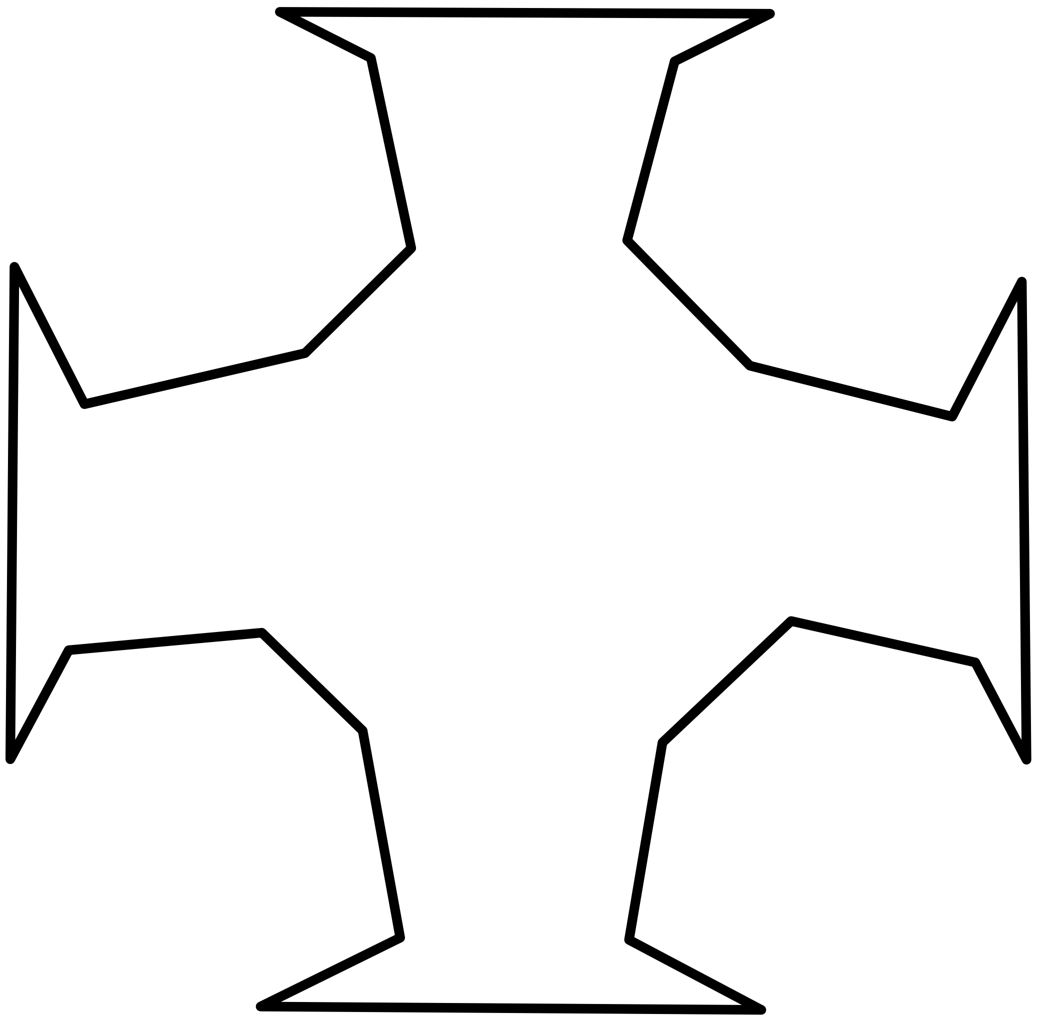

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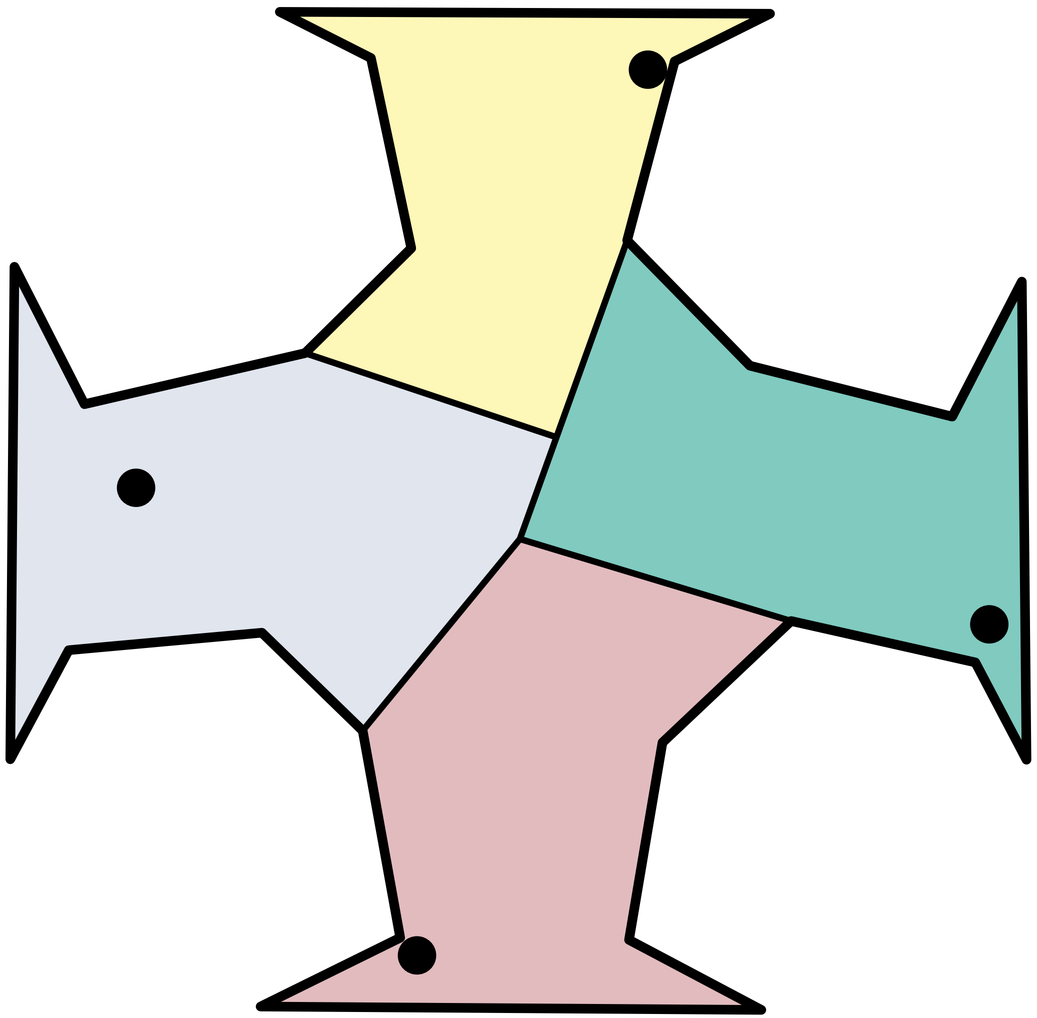

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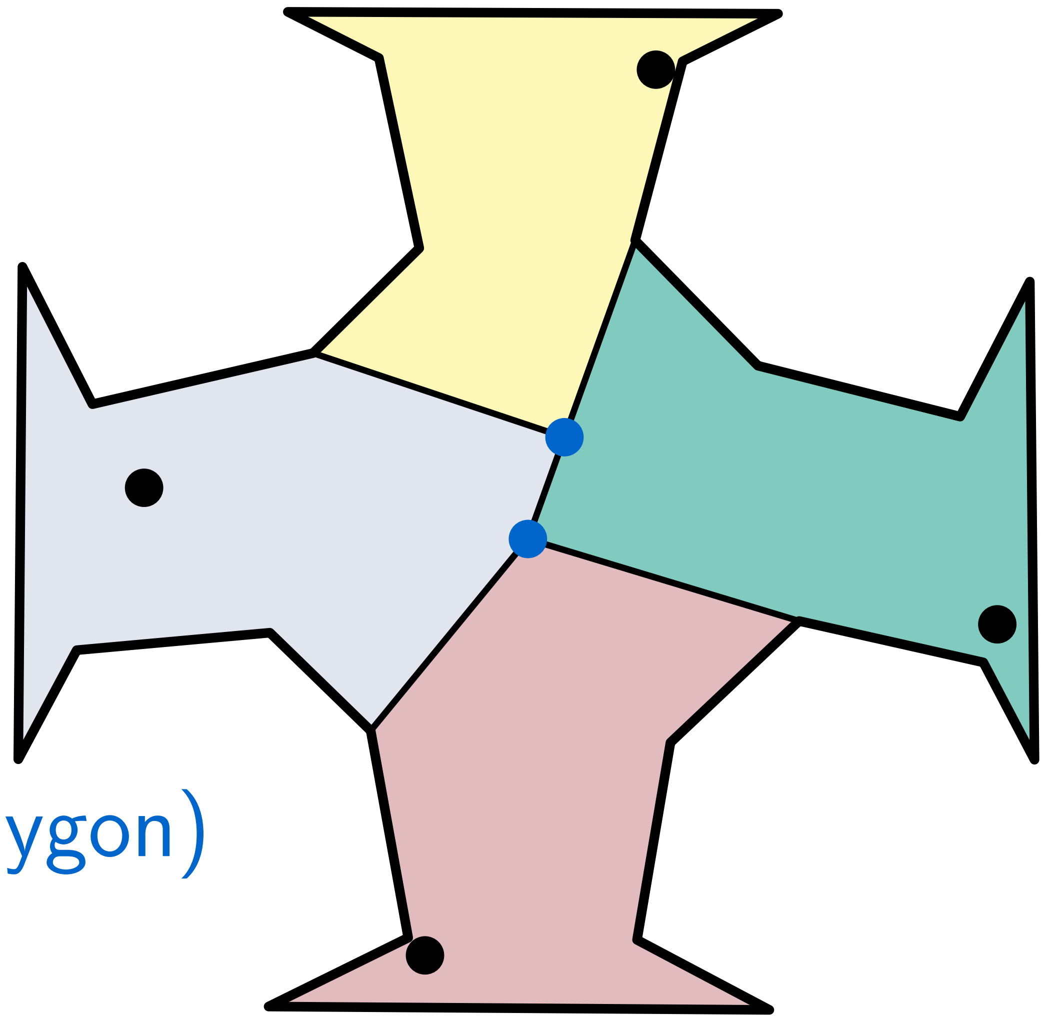

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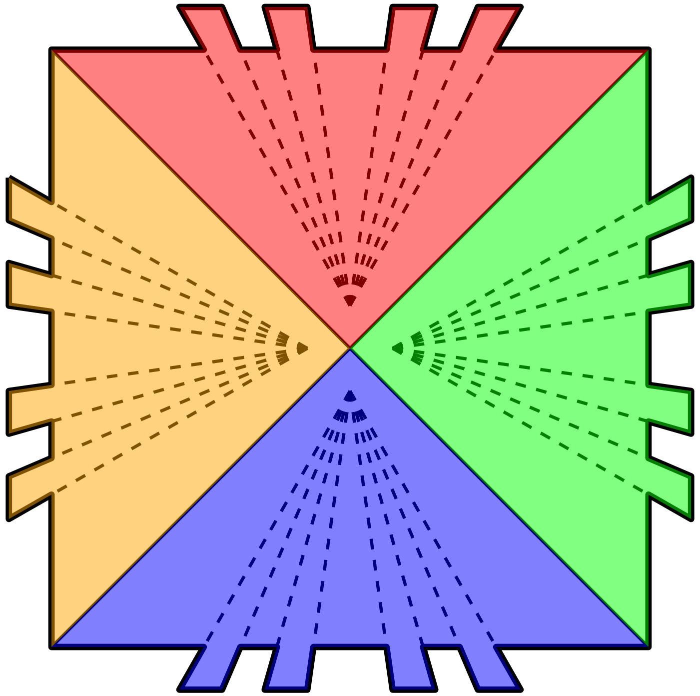

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(corner of the solution but not of input polygon)

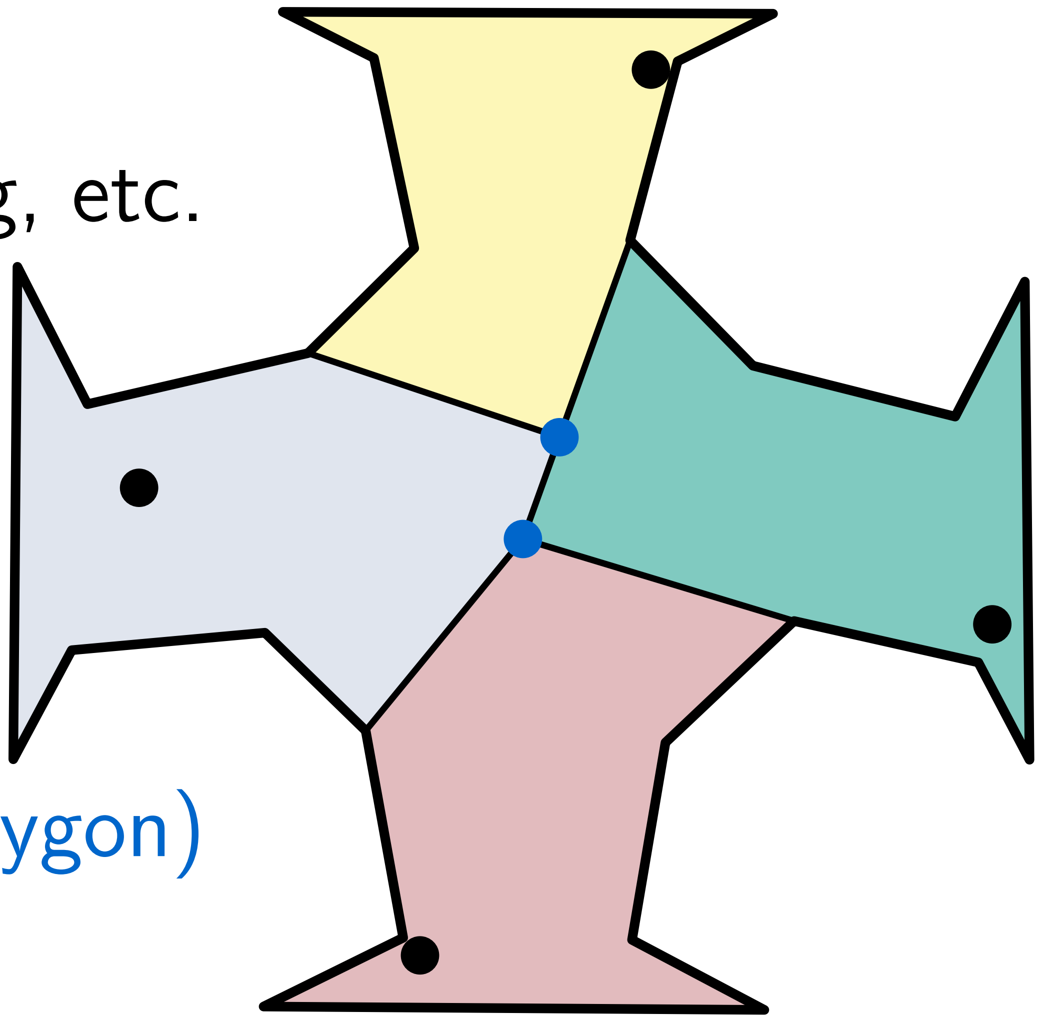
Without Steiner Points: $\tilde{O}(n^7)$ [Kei'85]

With Steiner points: $\text{ans} = 4$
Without Steiner points: $\text{ans} = \Omega(n)$

Star Partitions of Polygons

Given polygon (n = #corners),
Partition into few star pieces

Applications in CNC milling, route planning, etc.
Open for > 40 years if in P (or even NP)



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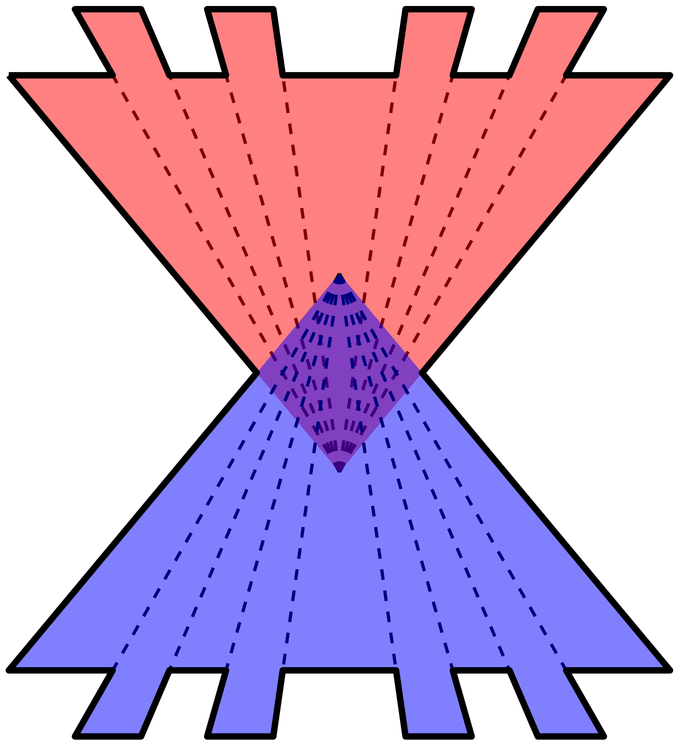
Art Gallery Problem

Related Problem: **Cover** vs **Partition**



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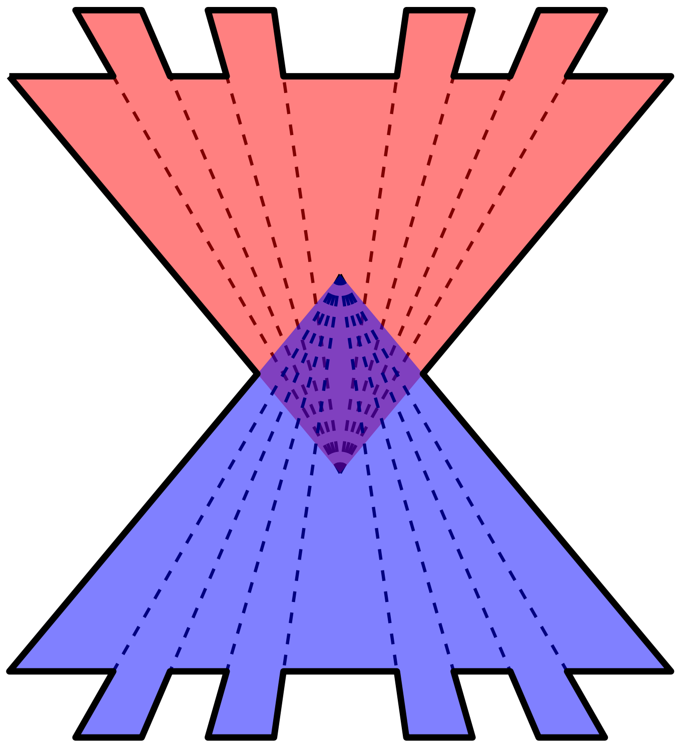
Related Problem: **Cover** vs **Partition**

Cover: Pieces can overlap ans=**2**

Partition: Pieces cannot overlap ans=$\Omega(n)$



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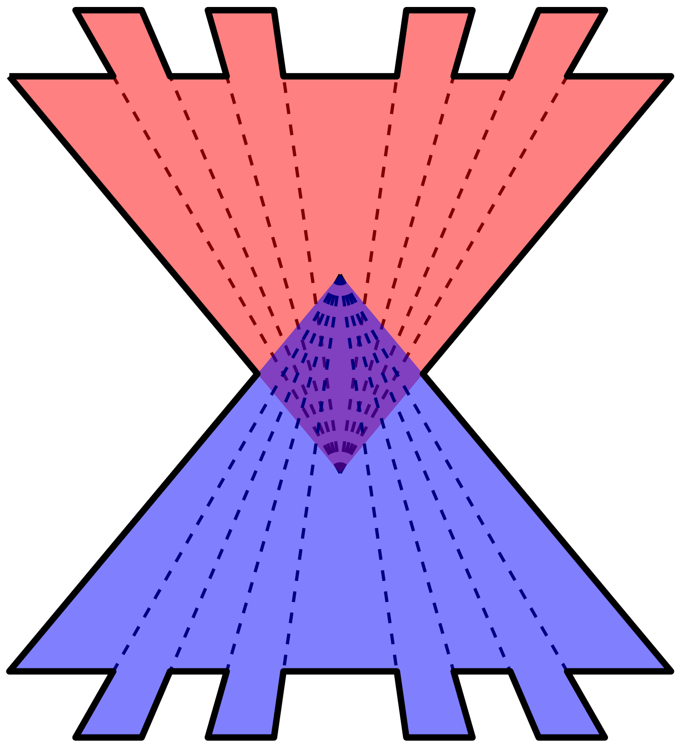
Theorem: (Cover)
Art Gallery Problem is $\exists\mathbb{R}$-complete [AAM'22]
(i.e., probably not even in NP)



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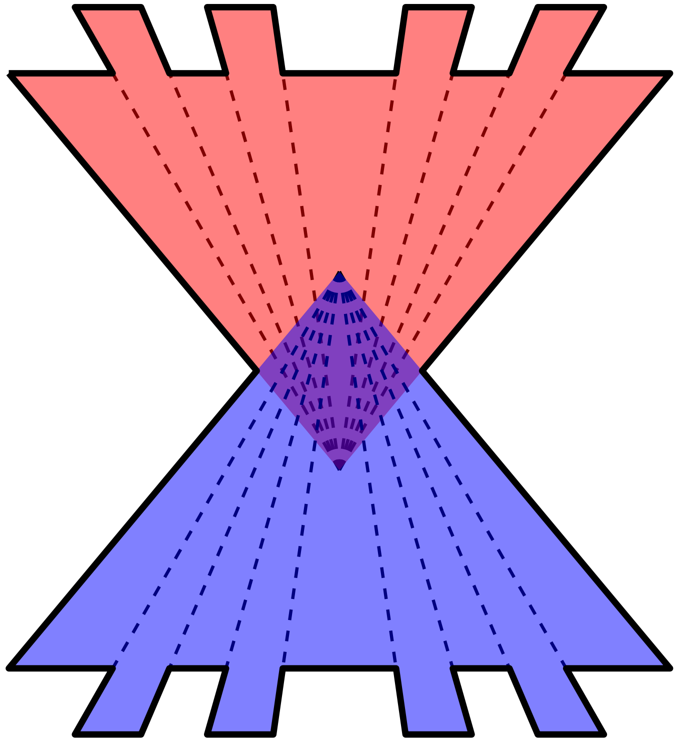

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Main Result

Our Main Result:

Minimum Star Partition of Simple Polygons in $O(n^{107})$ time

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Polynomial!

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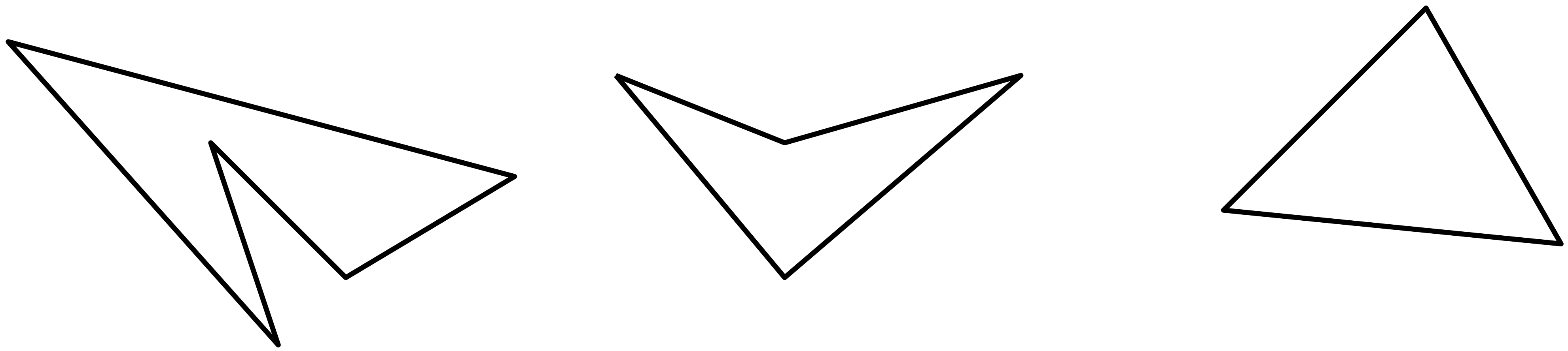
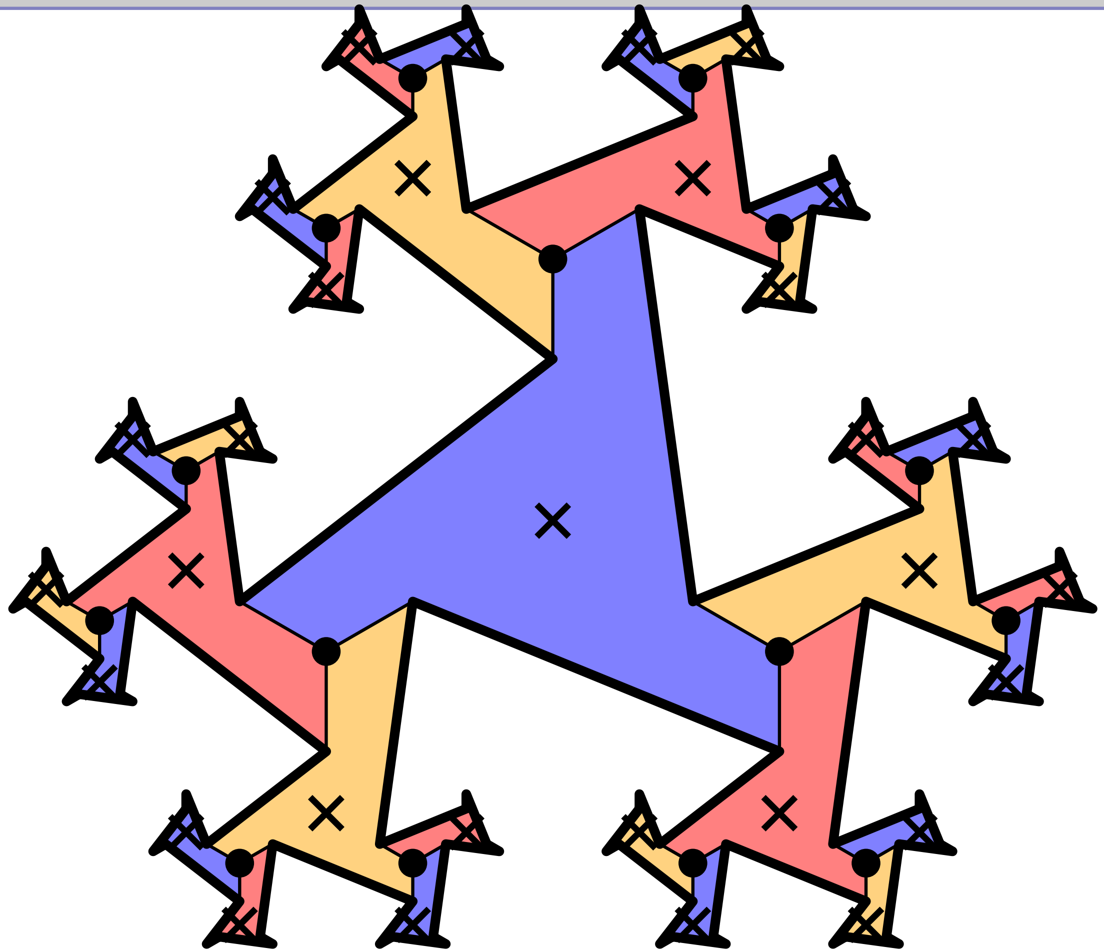
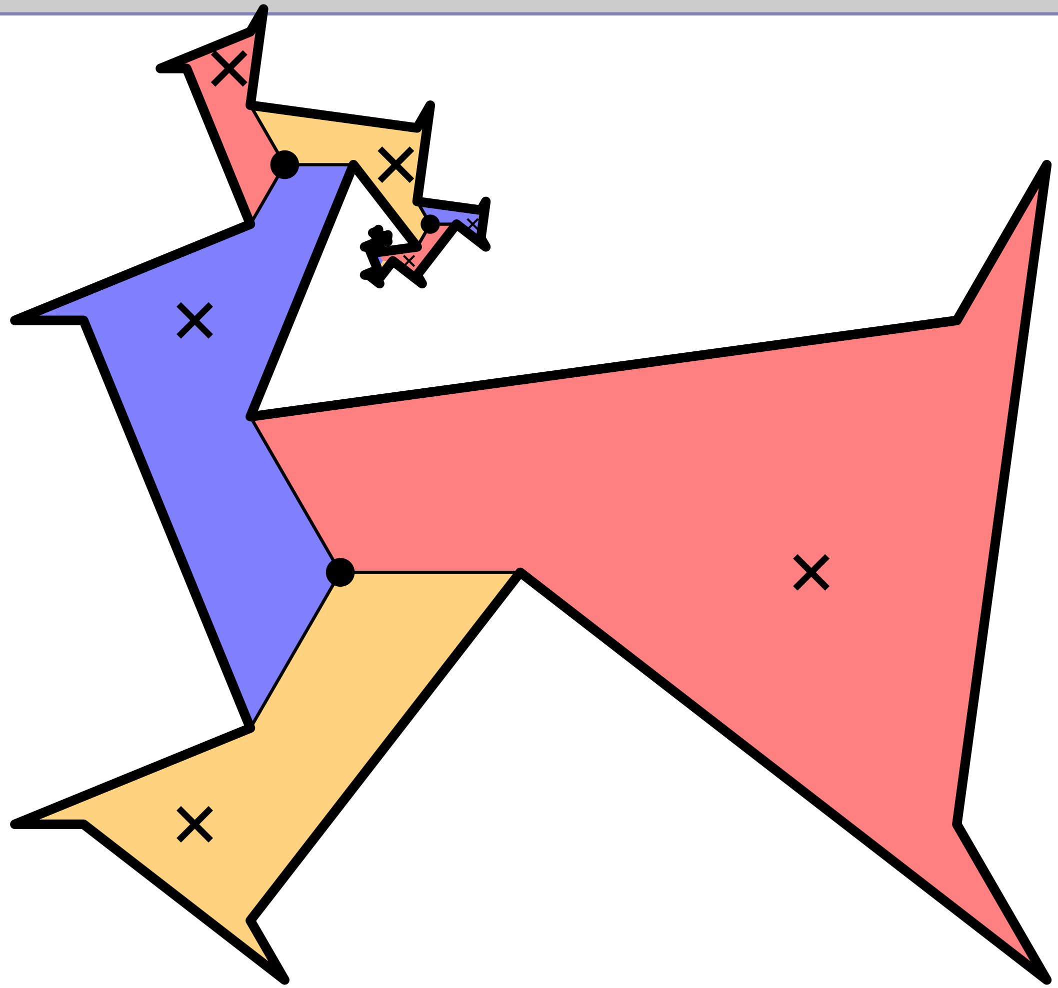
Polynomial!

Works in $\ll 1$ sec for $n \leq 5$, but $n = 6$ it is a bit slow...



Tricky Instances





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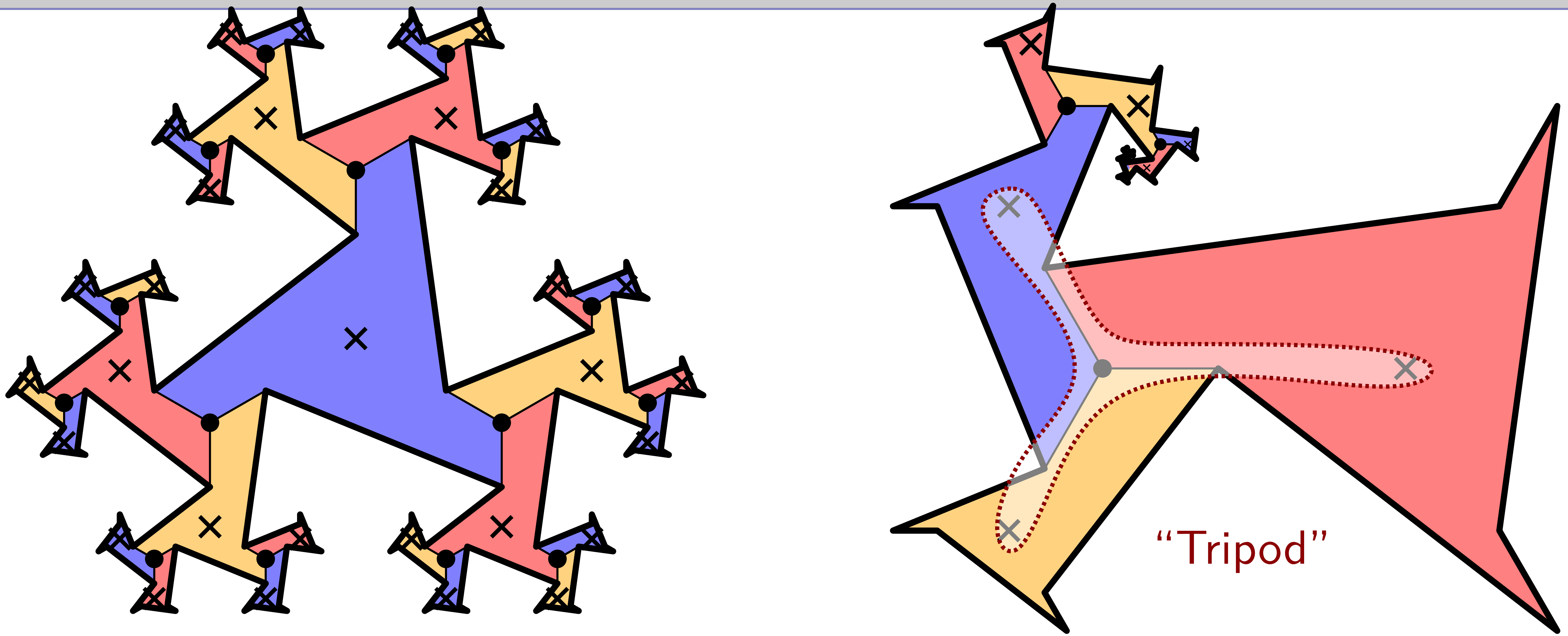
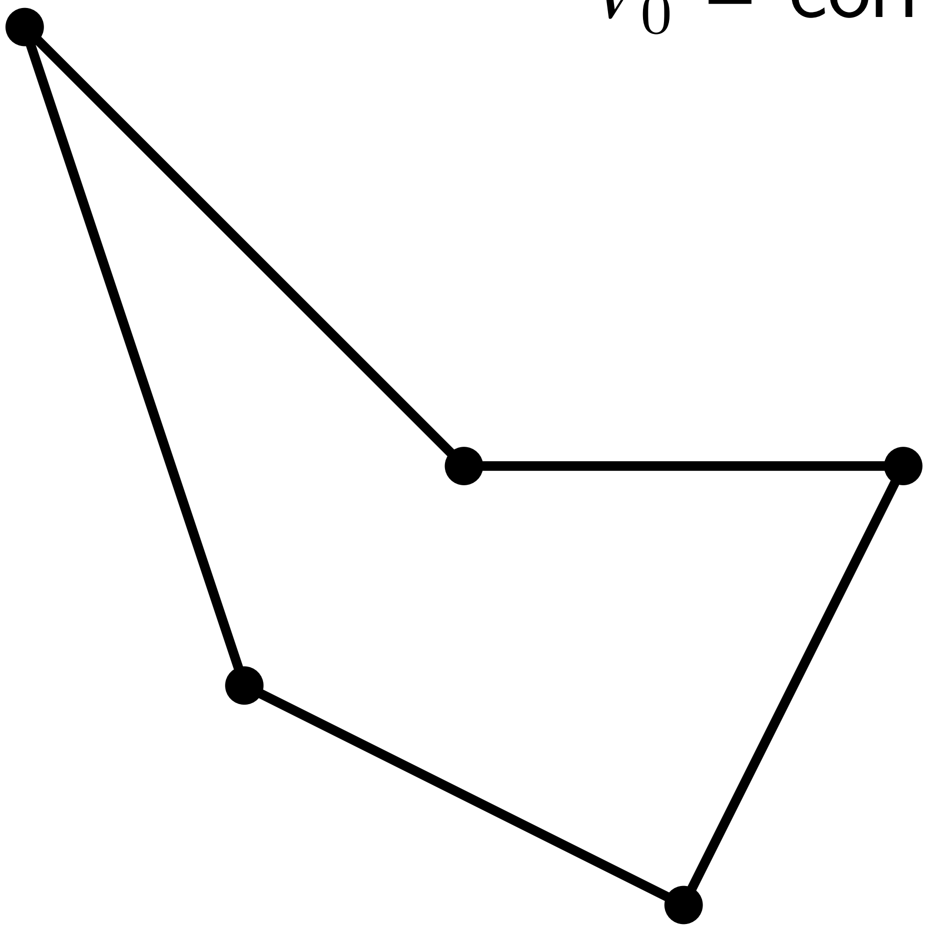


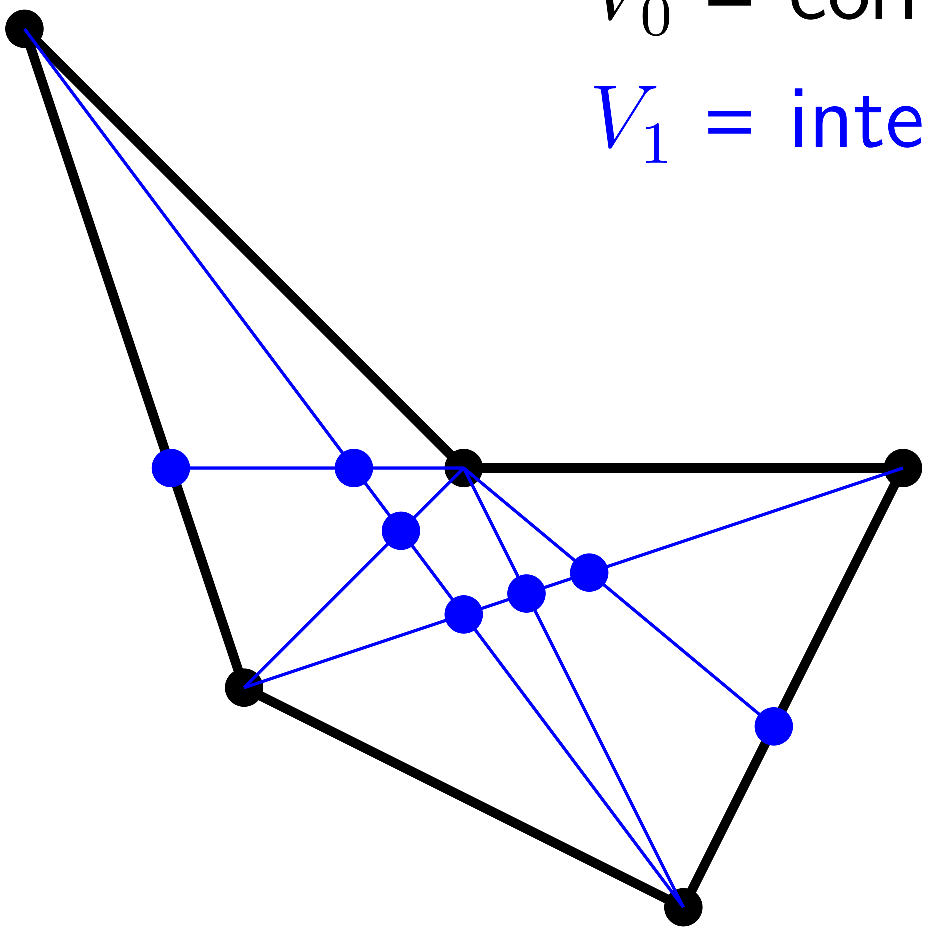
Tricky Instances

V_0 = corners of P



$|V_0| = n$

Tricky Instances



V_0 = corners of P

V_1 = intersections of lines between points in V_0

$|V_0| = n$ $|V_1| = O(n^4)$

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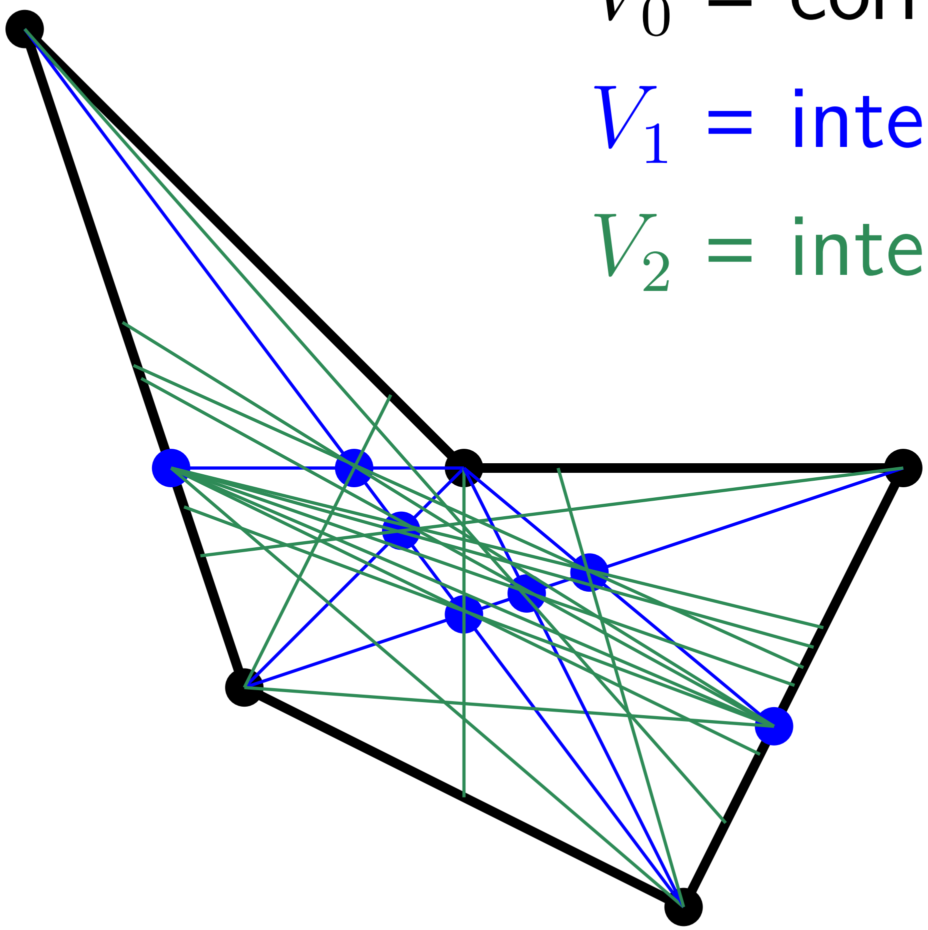

V_0 = corners of P

V_1 = intersections of lines between points in V_0

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$|V_0| = n$ $\quad |V_1| = O(n^4)$ $\quad |V_2| = O(n^{16})$

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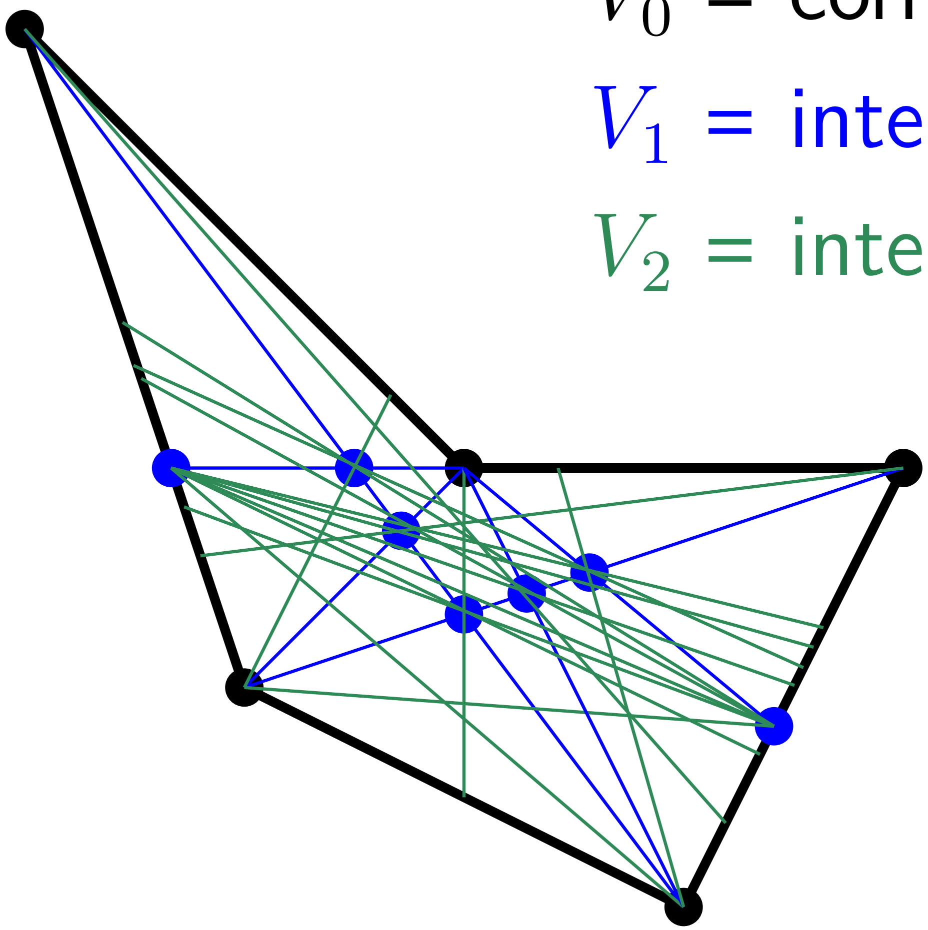

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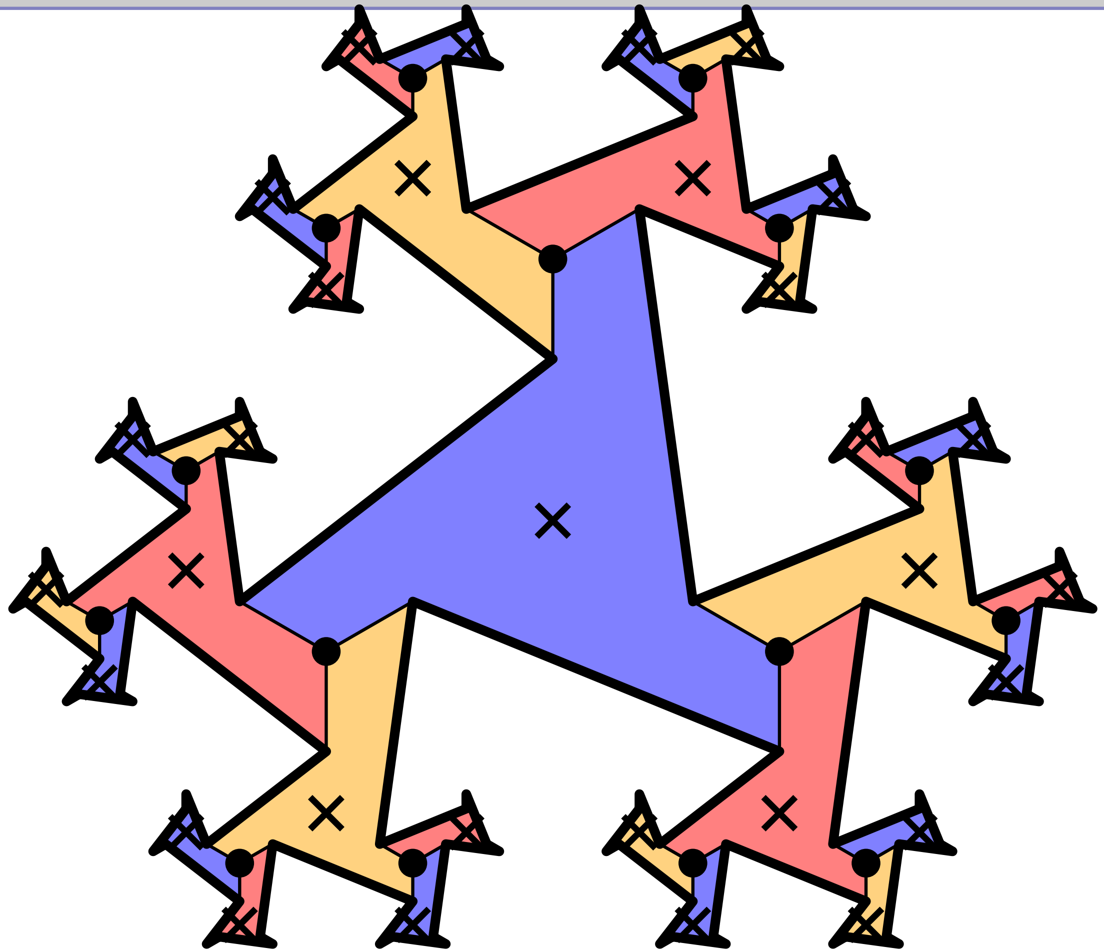
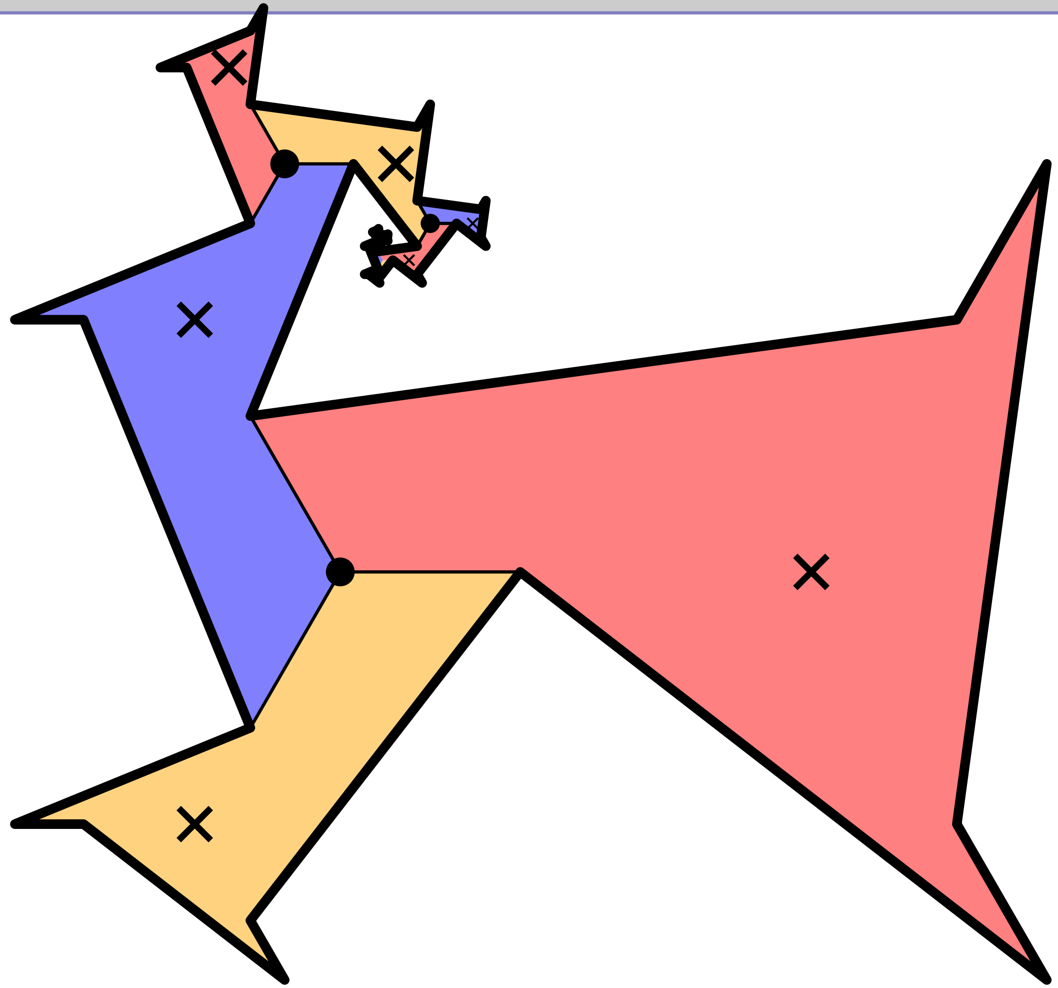
$|V_0| = n$ $|V_1| = O(n^4)$ $|V_2| = O(n^{16})$ $|V_k| = O(n^{(4^k)})$

Tricky Instances





Good News: Exists solution with all (Steiner) points in V_n (unlike covering!)

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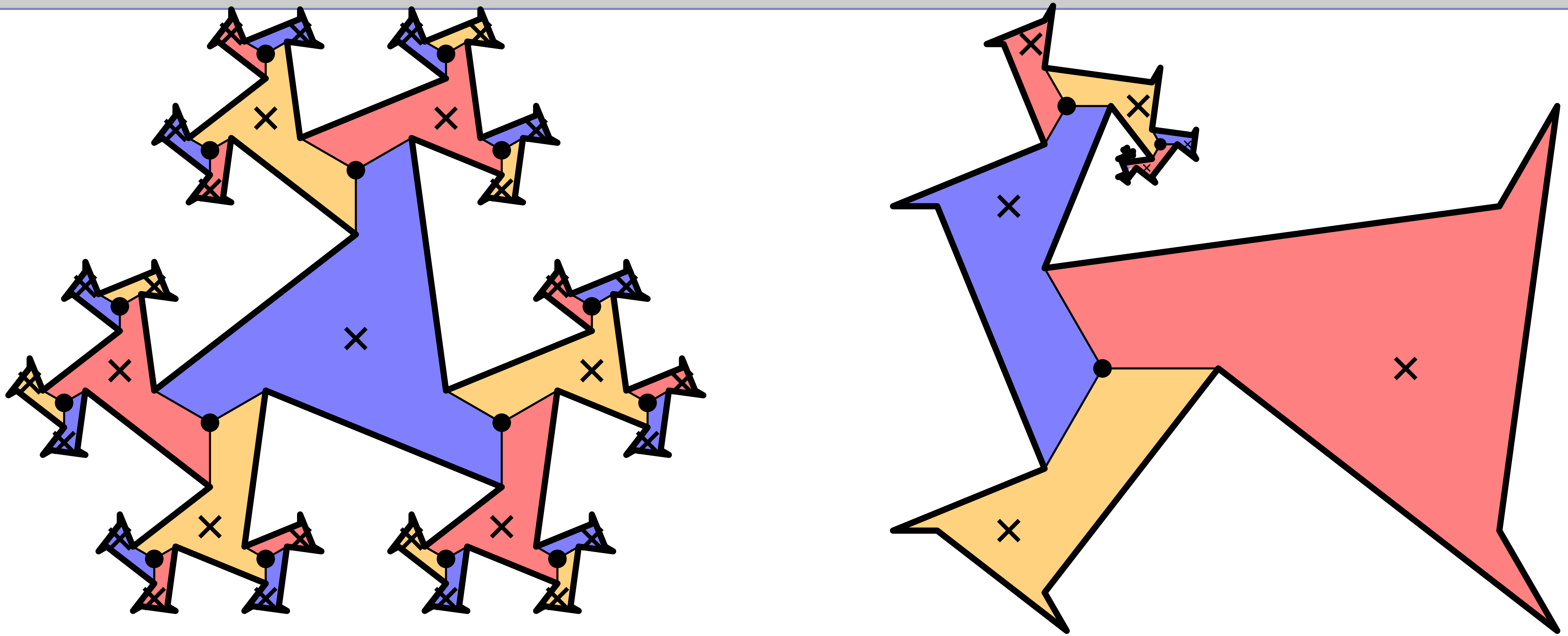

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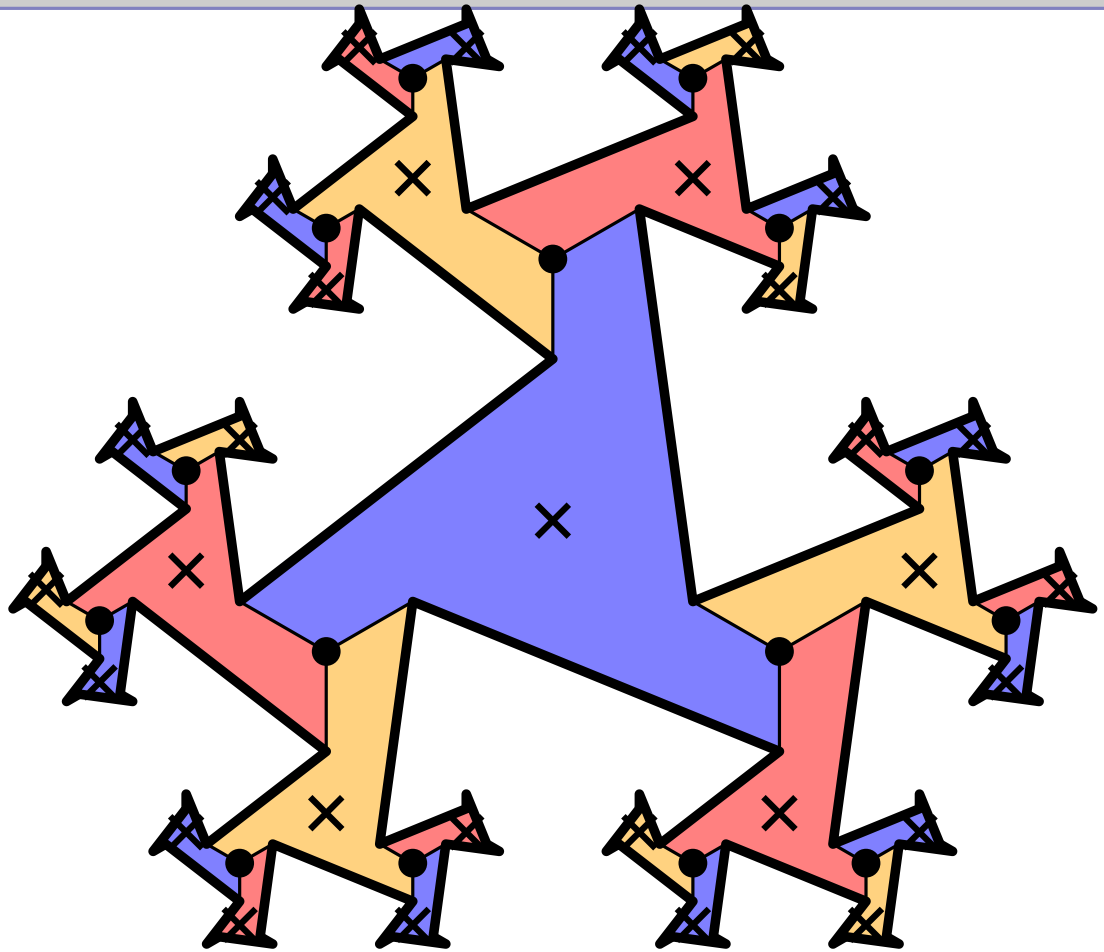

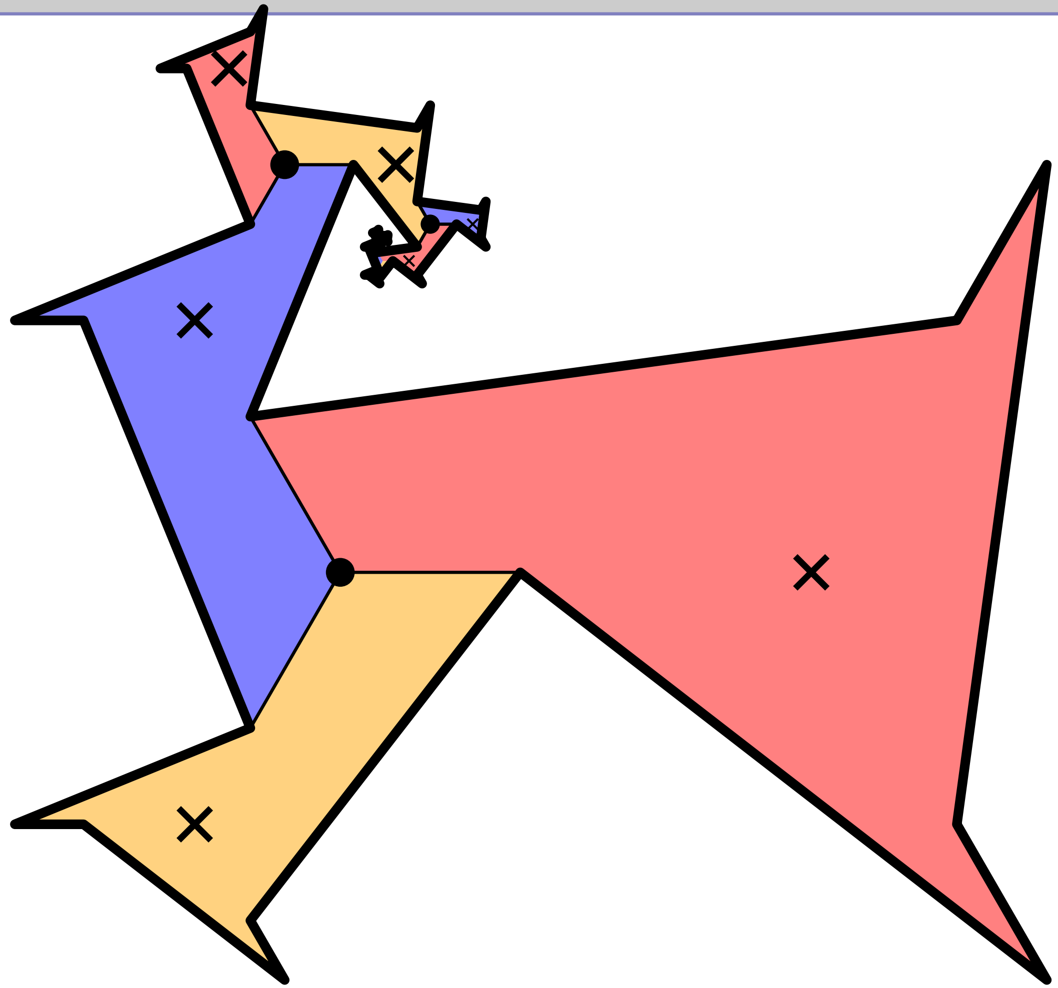

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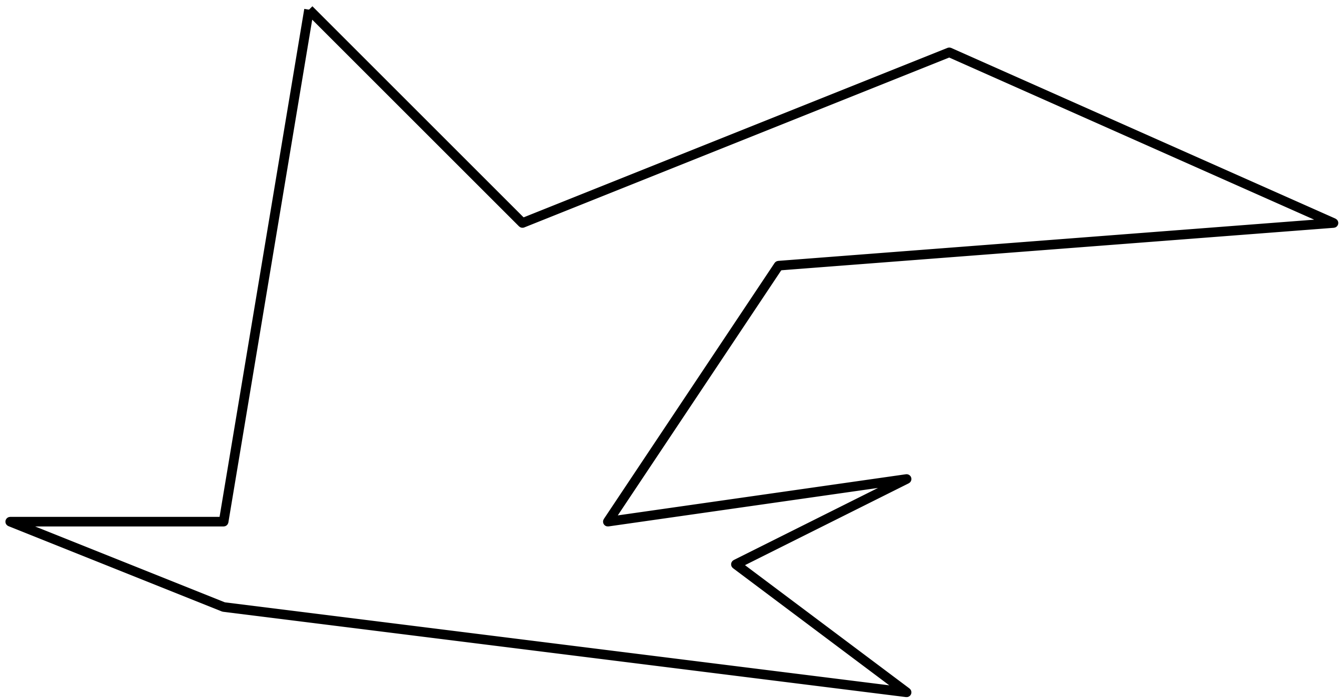
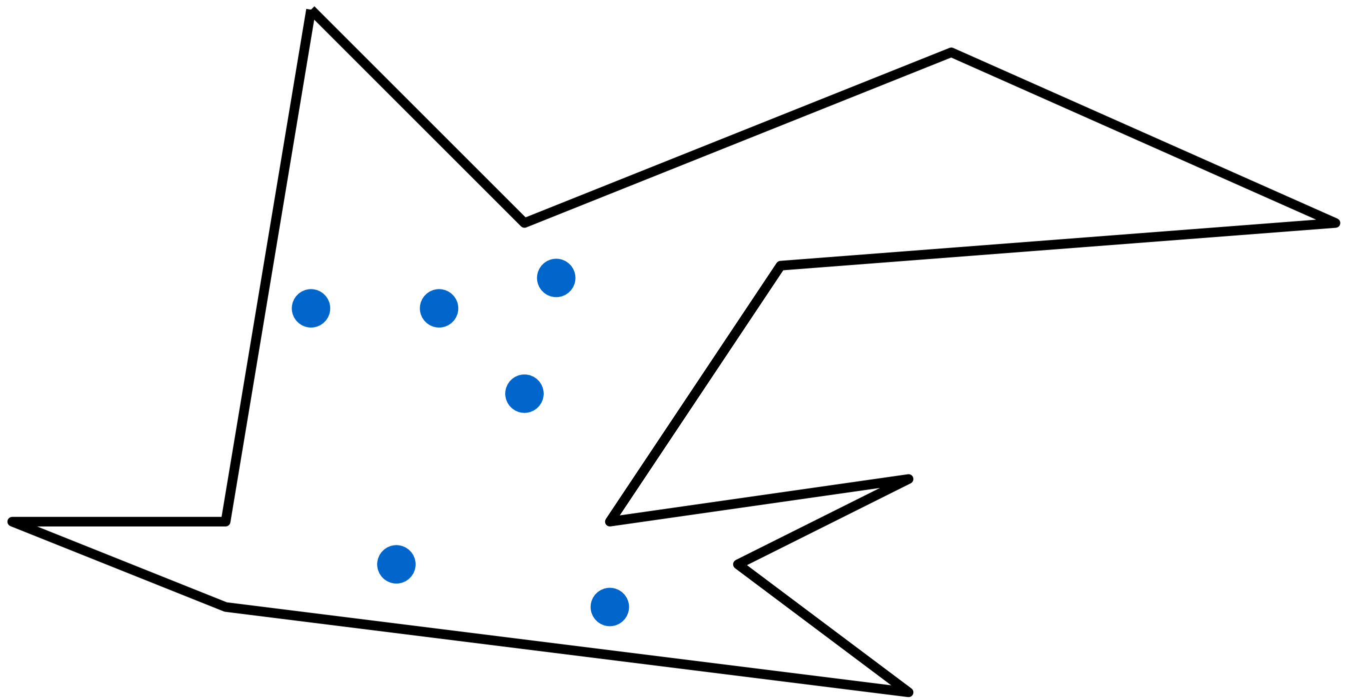
Good News: "Tripod" structure is only "tricky" case

Algorithm Outline



Algorithm Outline

1. Find small set of potential star-centers: S^{centers}



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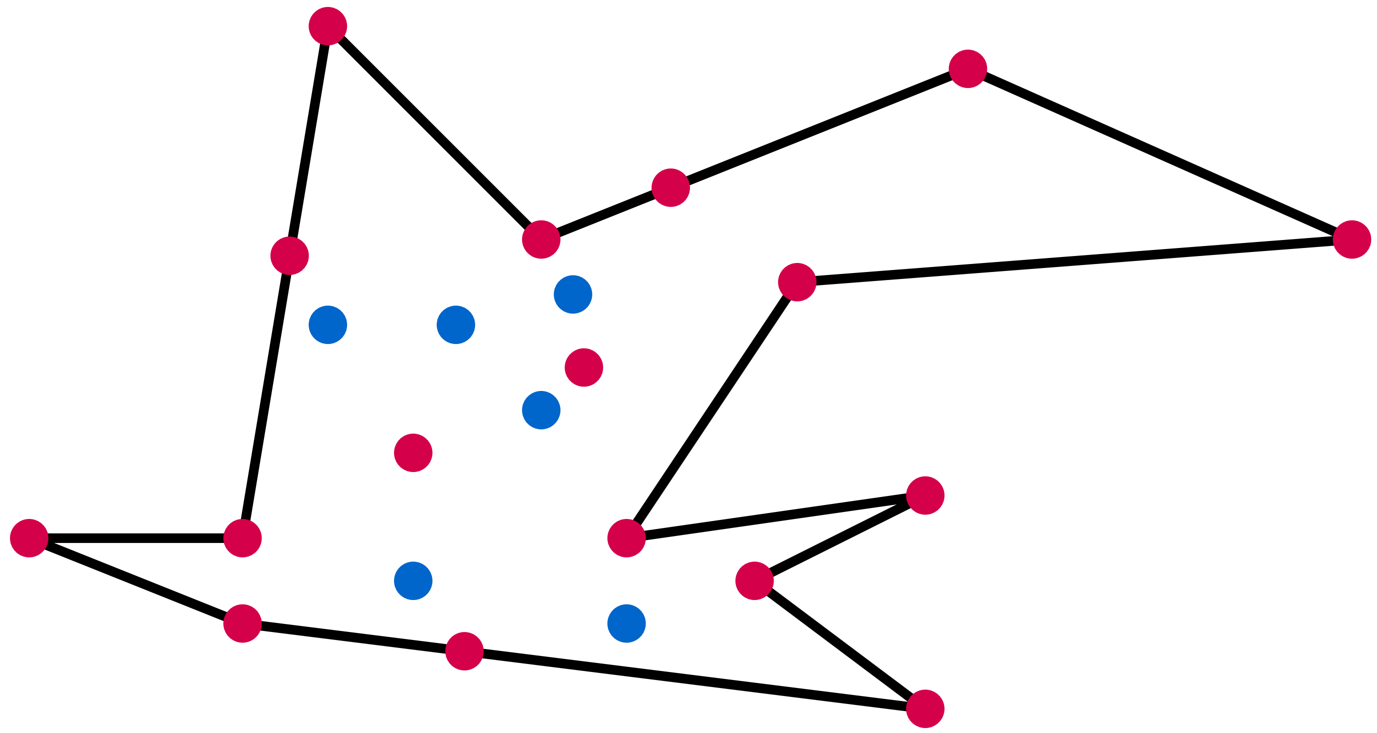

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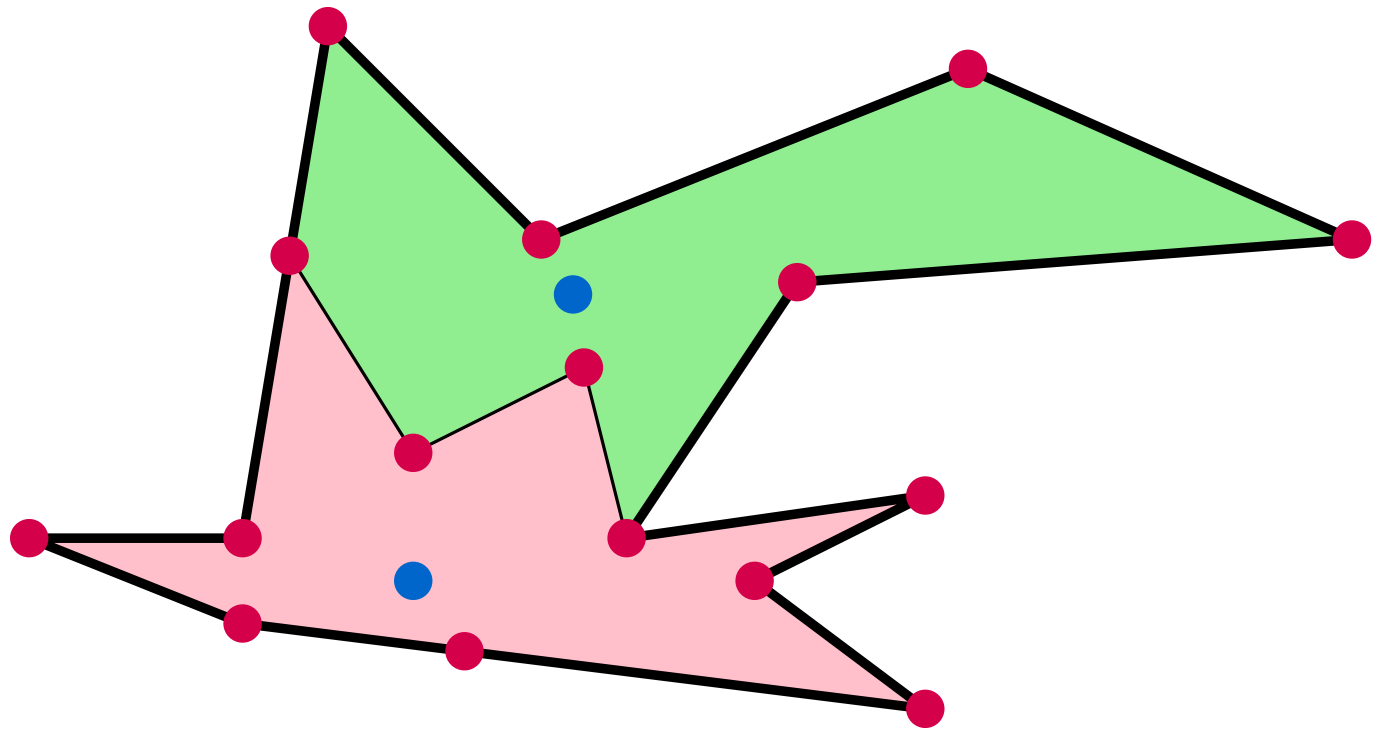

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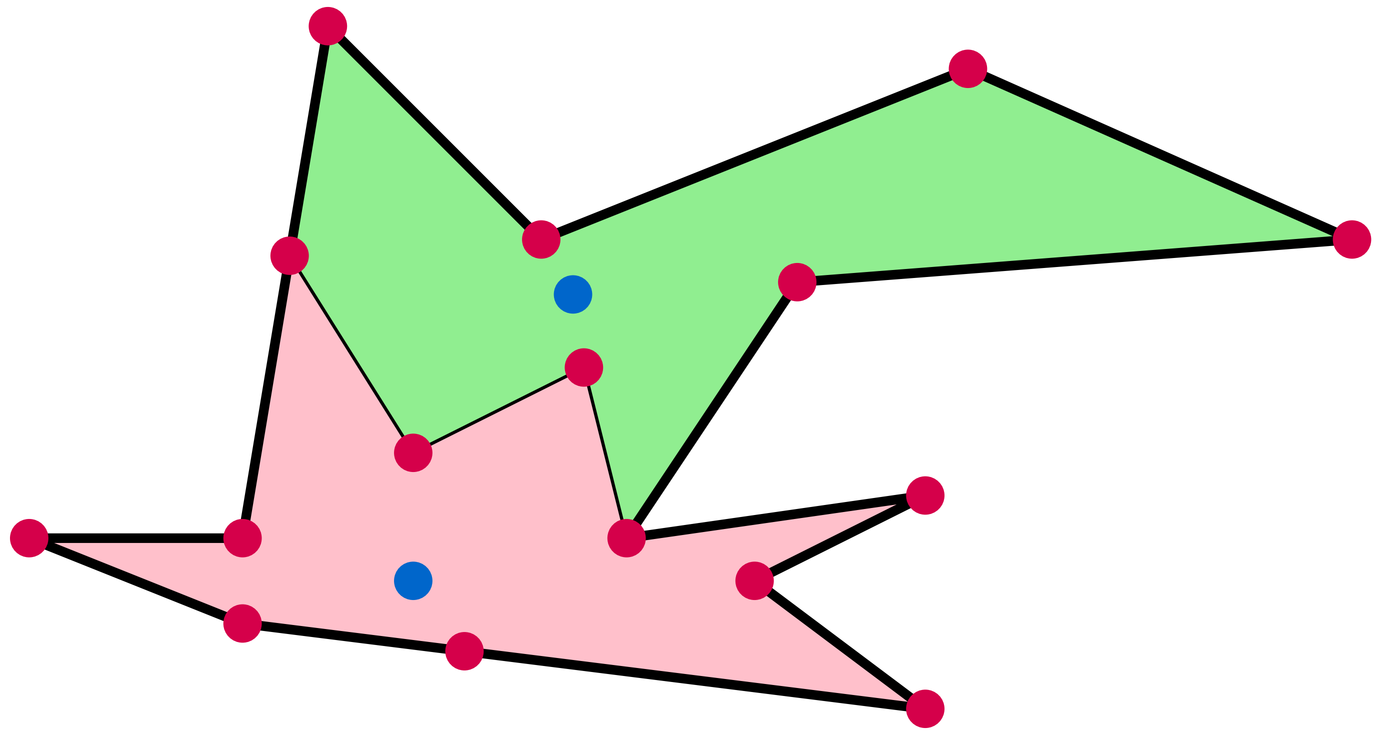

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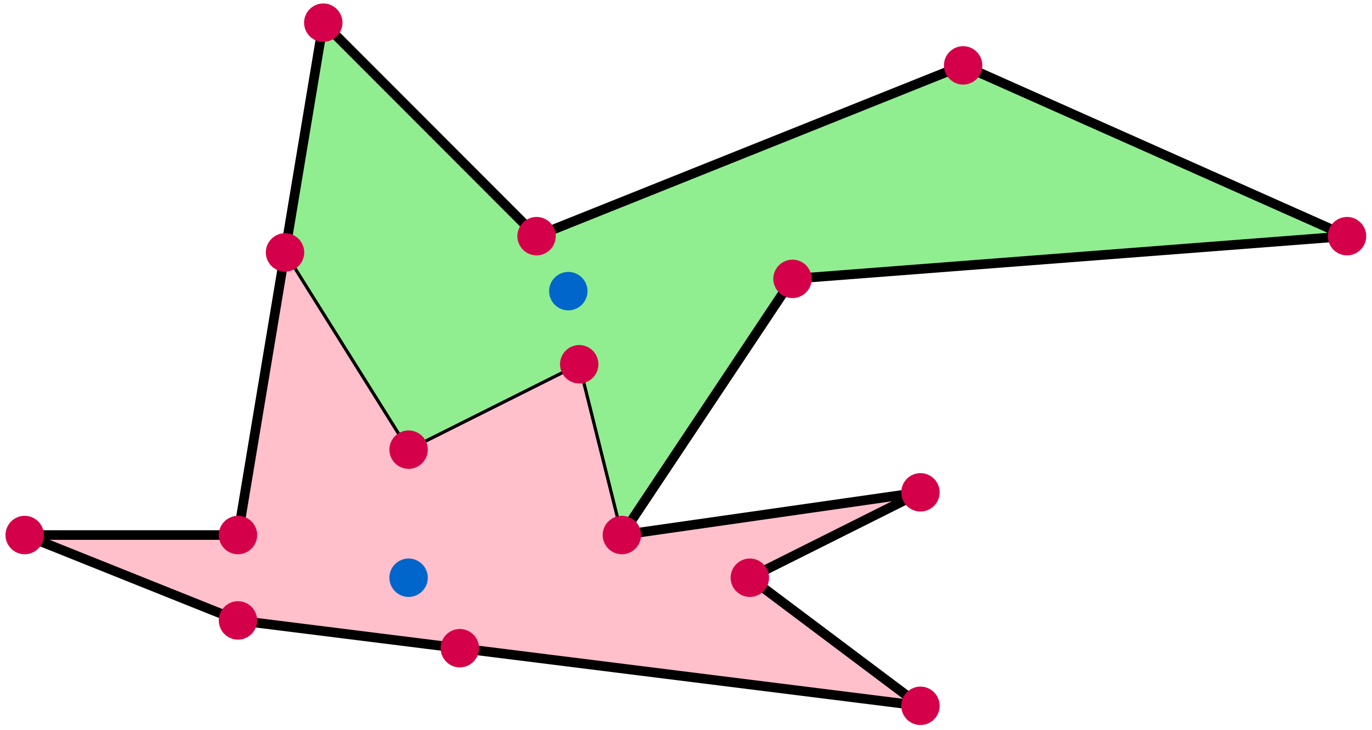
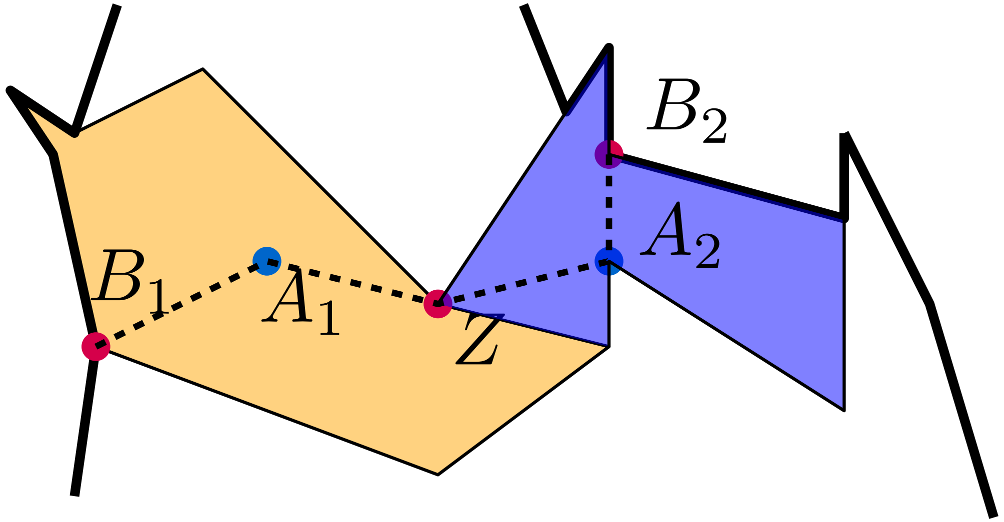
Kind of easy

Every piece touches the boundary





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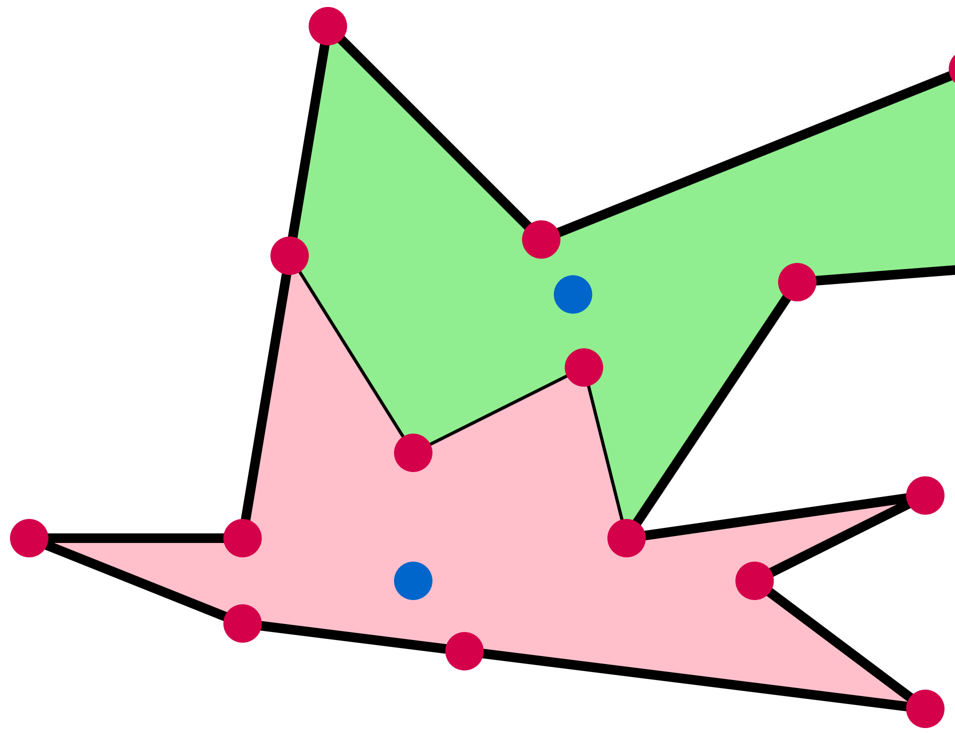

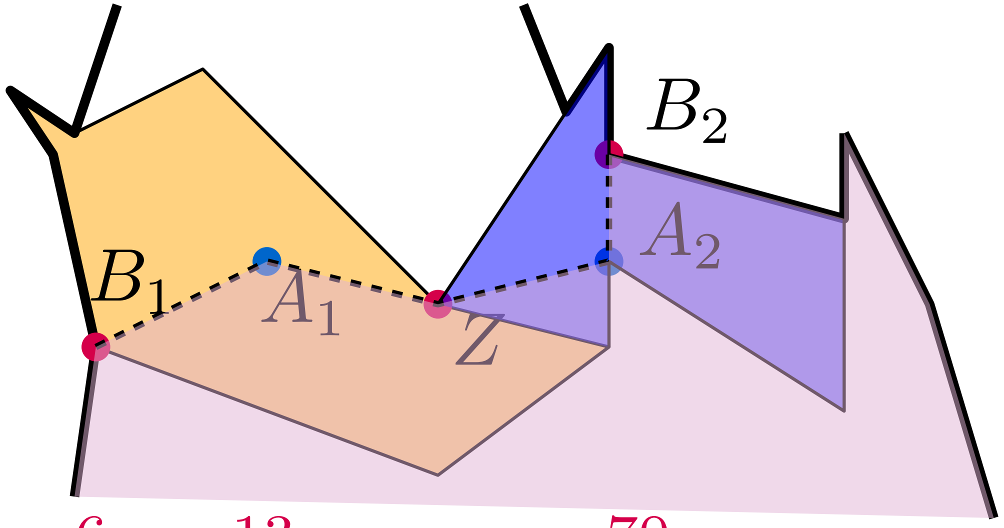

$O(n^{13} \cdot n^6 \cdot n^{32} \cdot n^6 \cdot n^{13}) = O(n^{70})$ states

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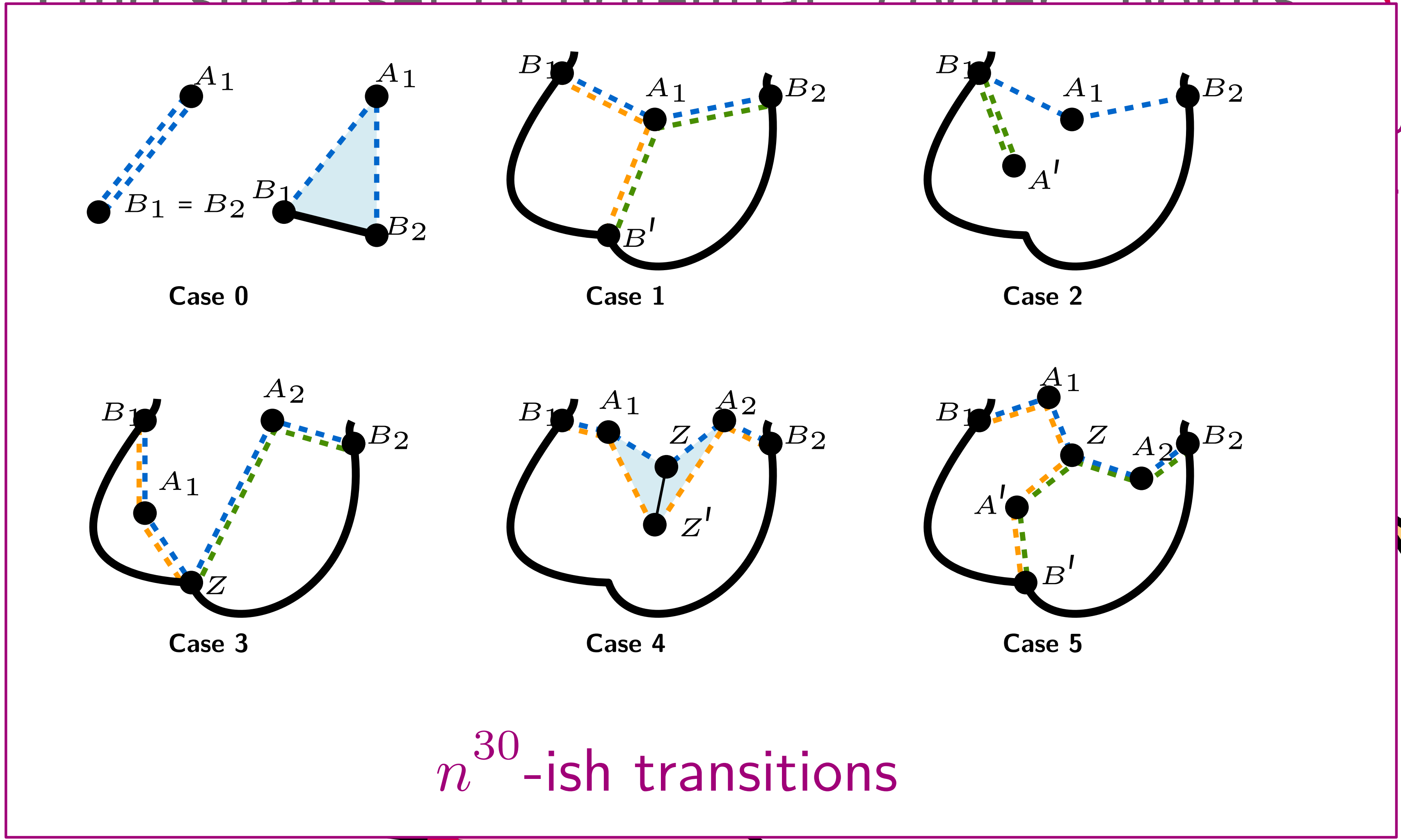

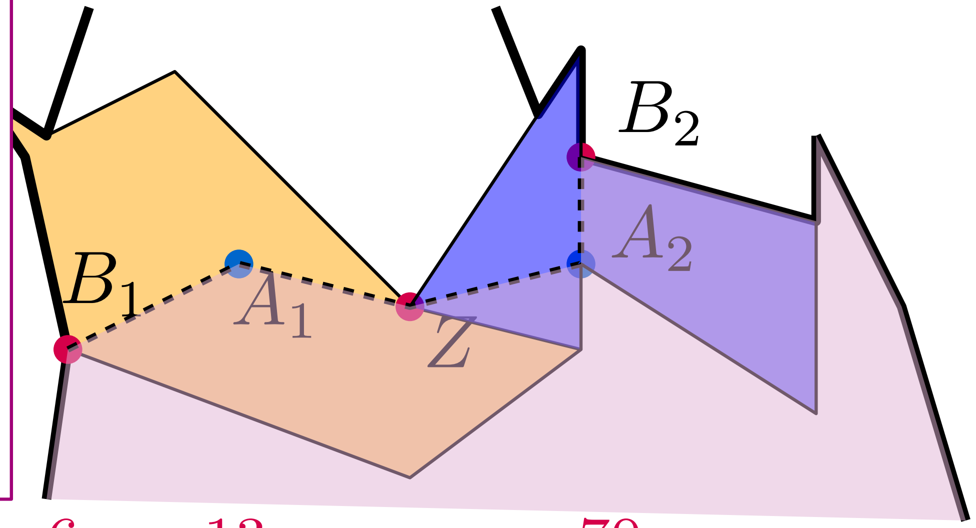

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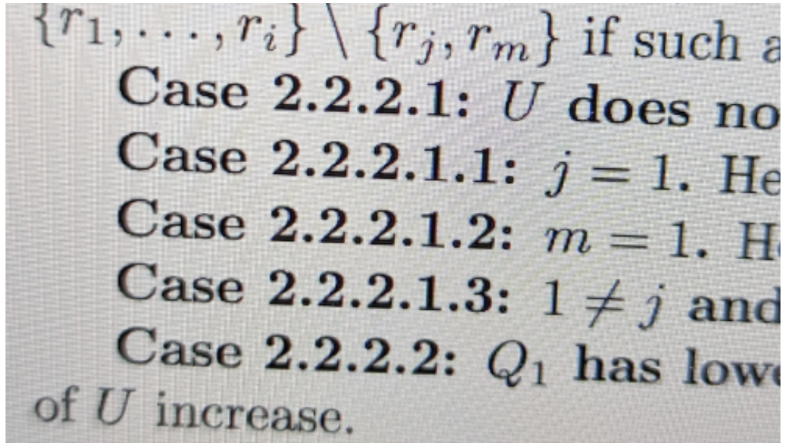
By case analysis...



$\{r_1, \ldots, r_i\} \setminus \{r_j, r_m\}$ if such a

Case 2.2.2.1: U does no

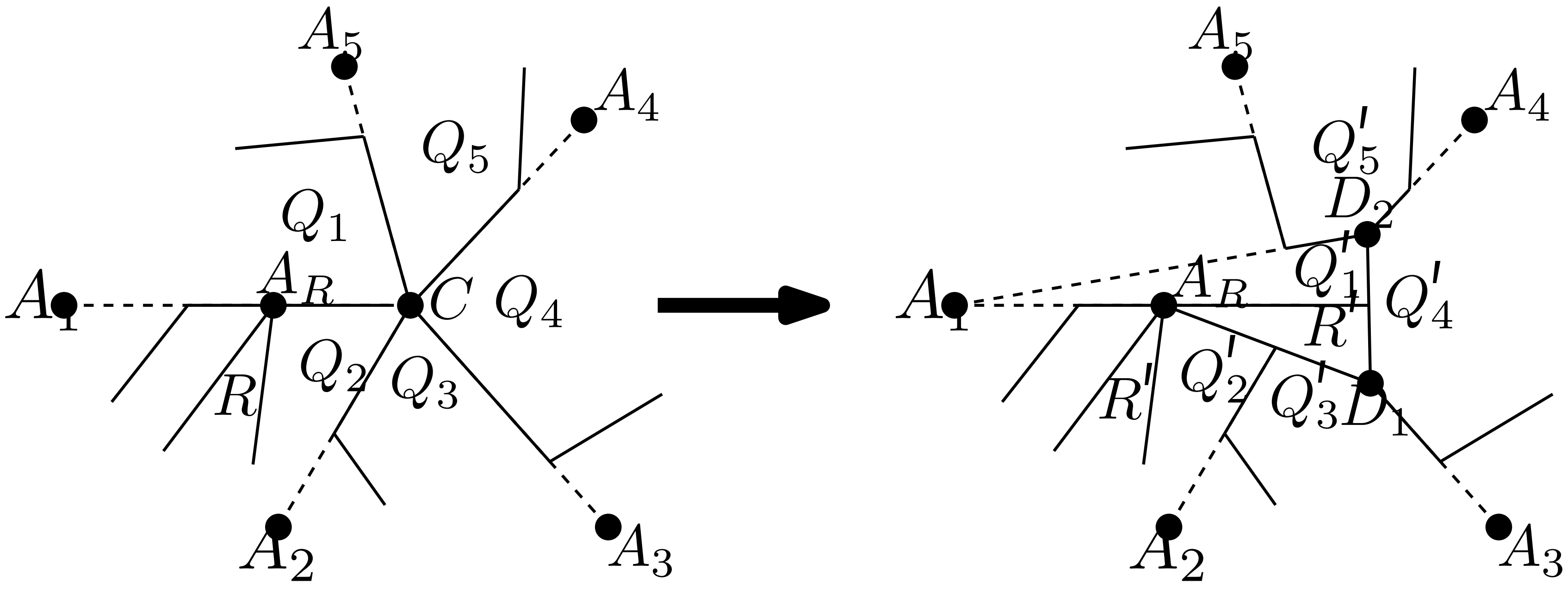
Case 2.2.2.1.1: $j = 1$. He

Case 2.2.2.1.2: $m = 1$. H

Case 2.2.2.1.3: $1 \neq j$ and

Case 2.2.2.2: Q_1 has low

of U increase.



Hard part

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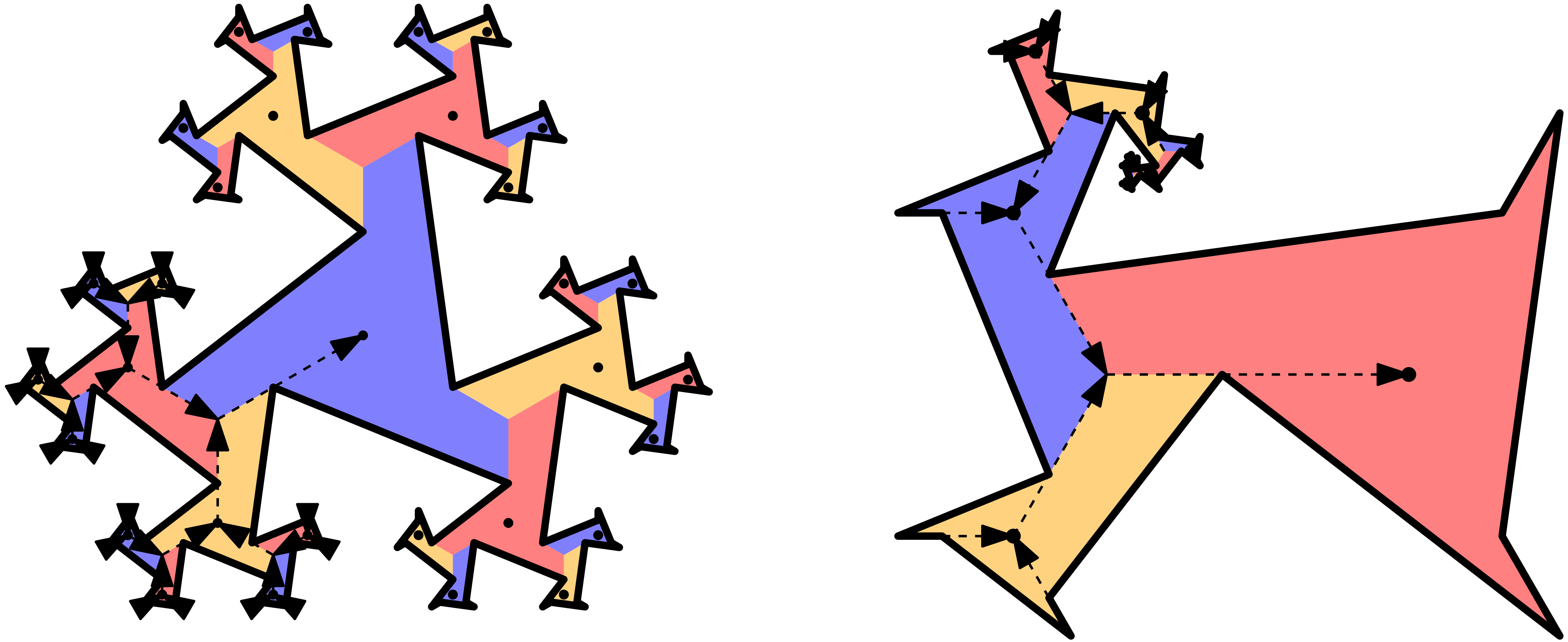

Ingredient 1: “Tripods” form rooted trees

Hard part

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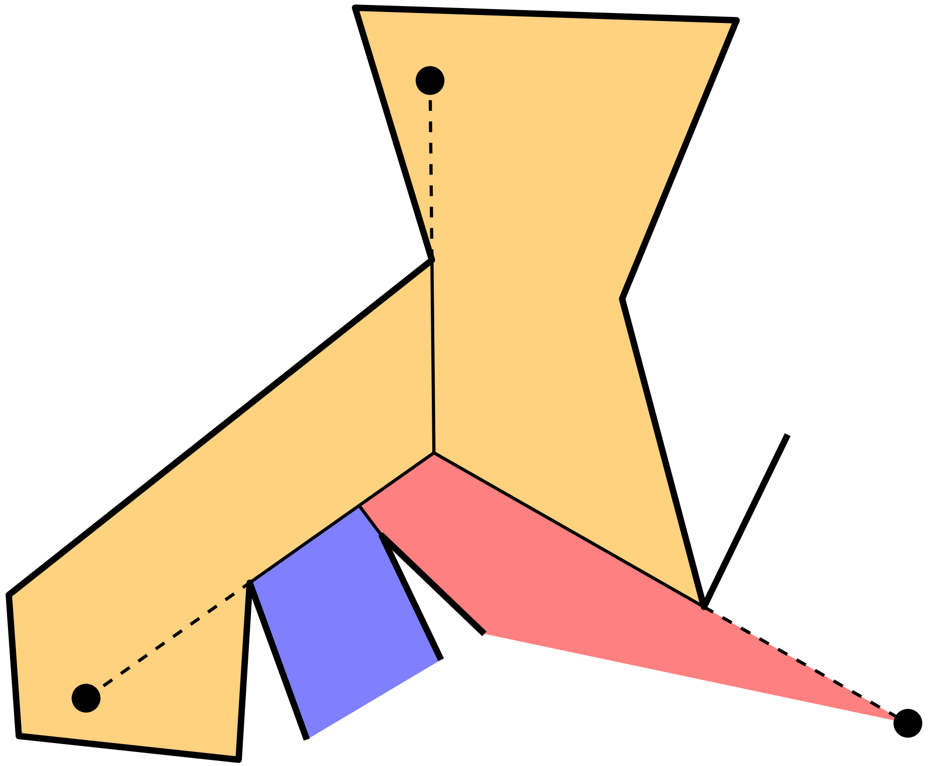

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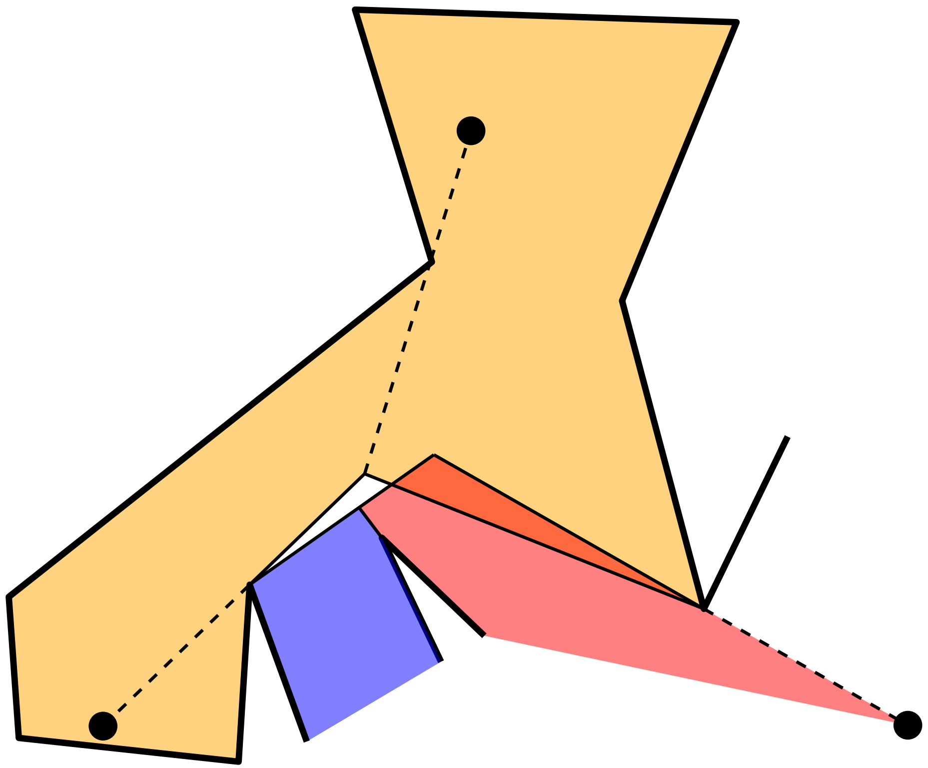

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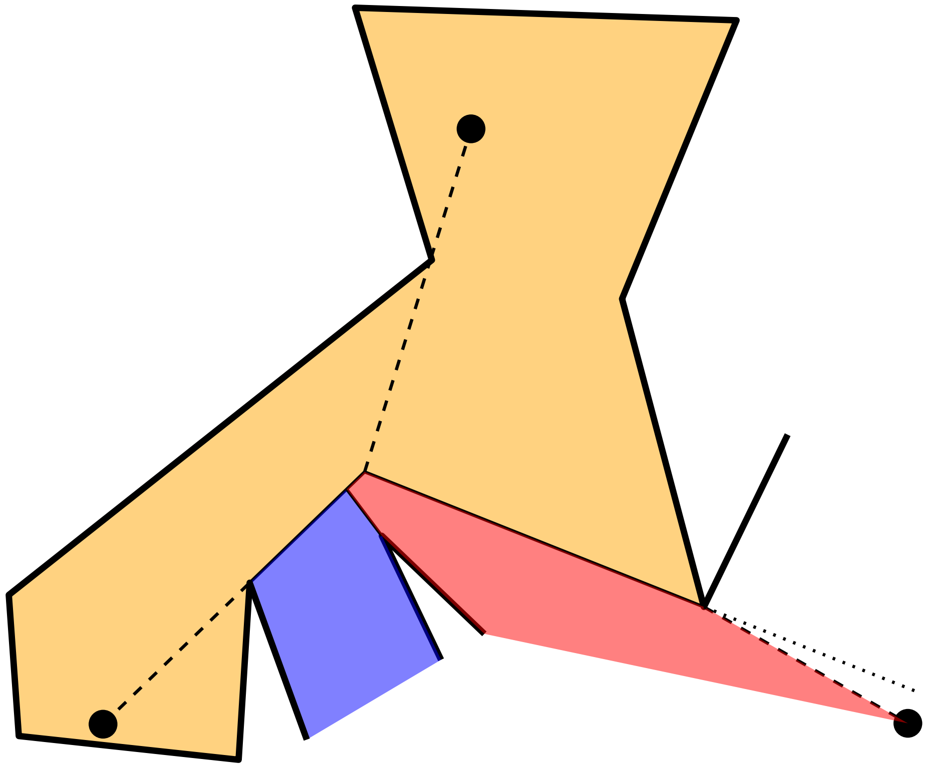

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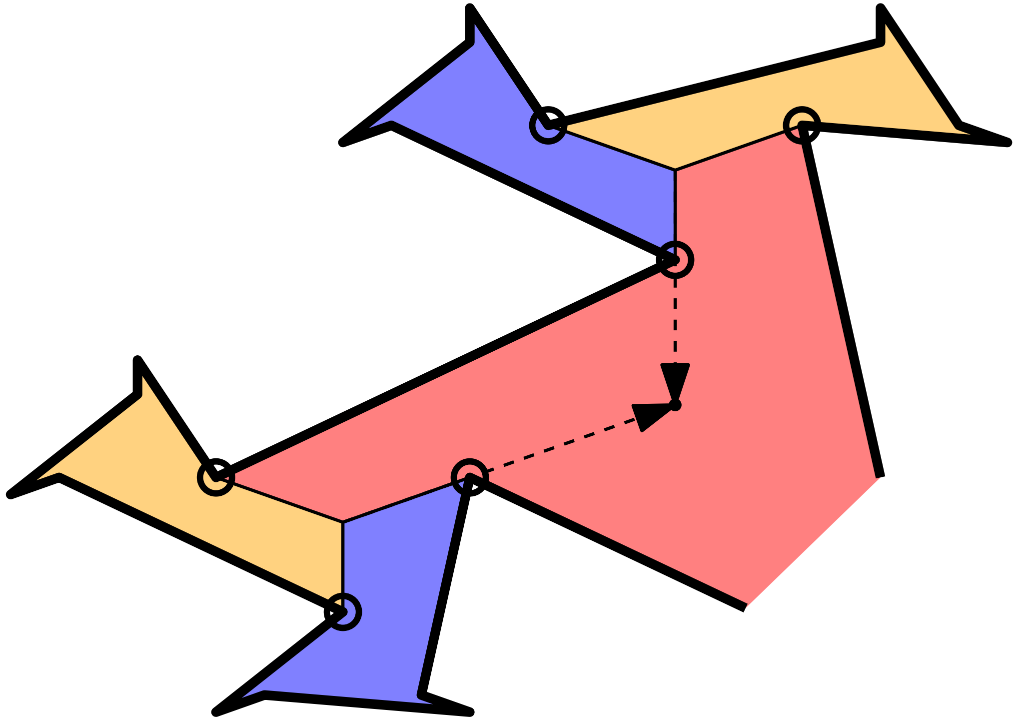

$O(n^6)$ candidate star-centers

Ingredient 1: “Tripods” form rooted trees

Ingredient 2: “Greedy Choice”

Ingredient 3: Bootstrap whole algorithm on smaller polygons

Summary

Our Main Result:

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Techniques:

Structural Properties of Optimial Solutions $+$ DP

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Minimum Star Partition of Simple Polygons in $O(n^{107})$ time

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Triangle Partition?

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Thanks!