Sublinear-Round Parallel Matroid Intersection

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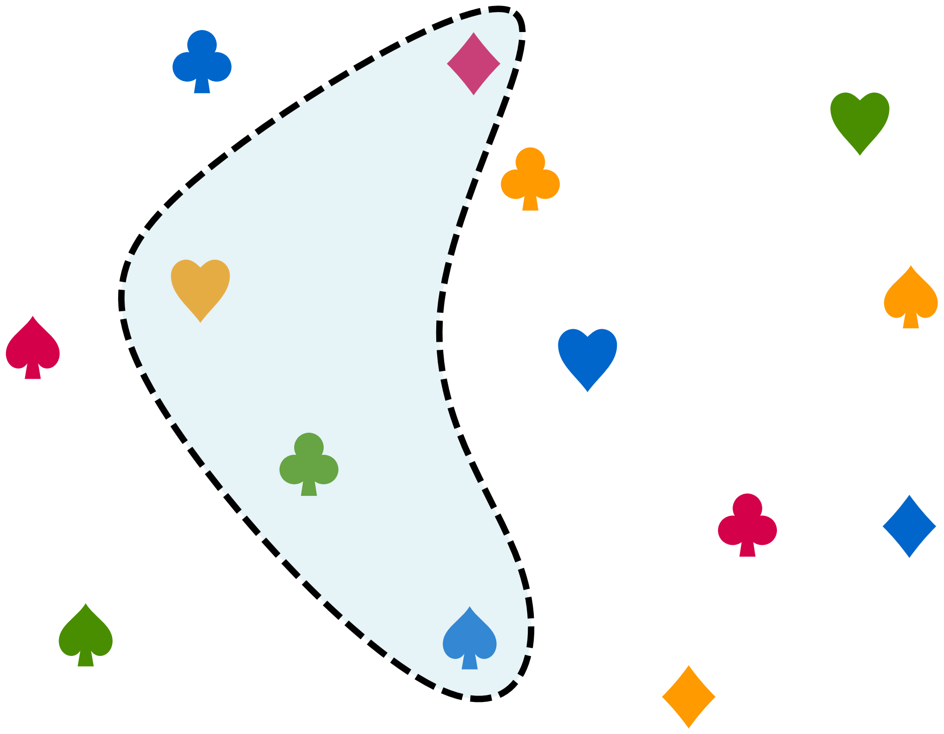
KTH Royal Institute of Technology

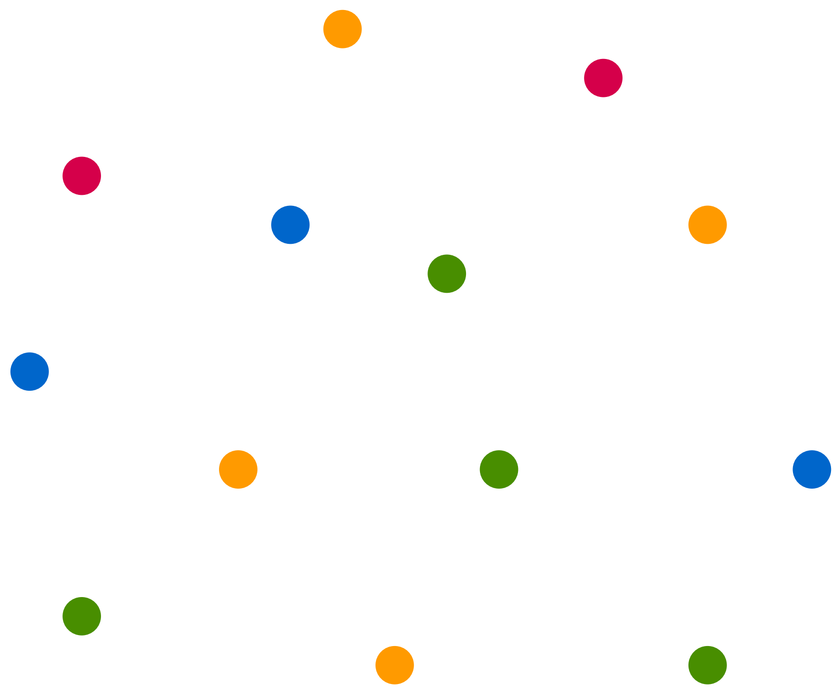
ICALP 2022



Matroids

Matroid $\mathcal{M} = (V, \mathcal{I})$

1. Ground set V of n elements



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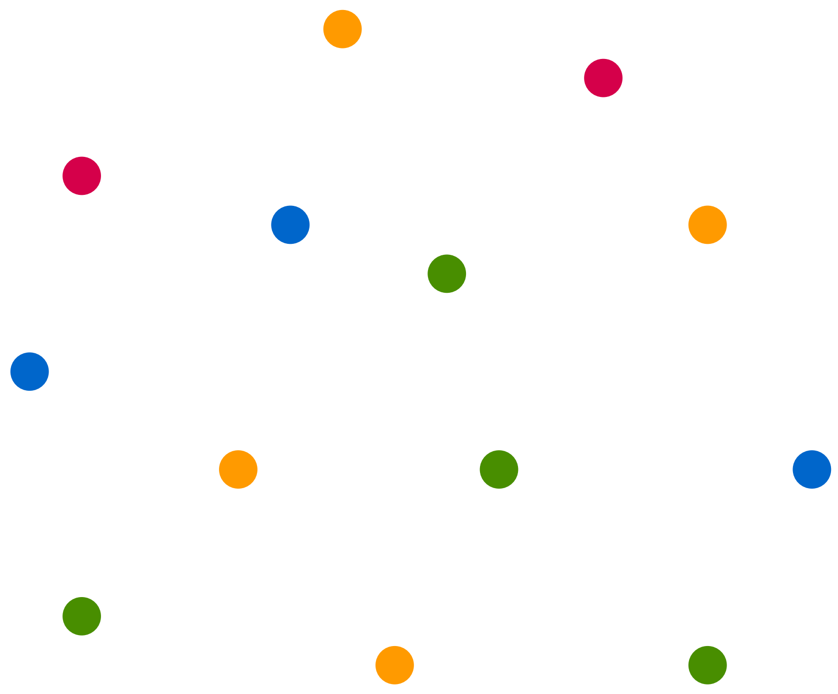

Eg. Colourful Matroid
"no duplicate colours"

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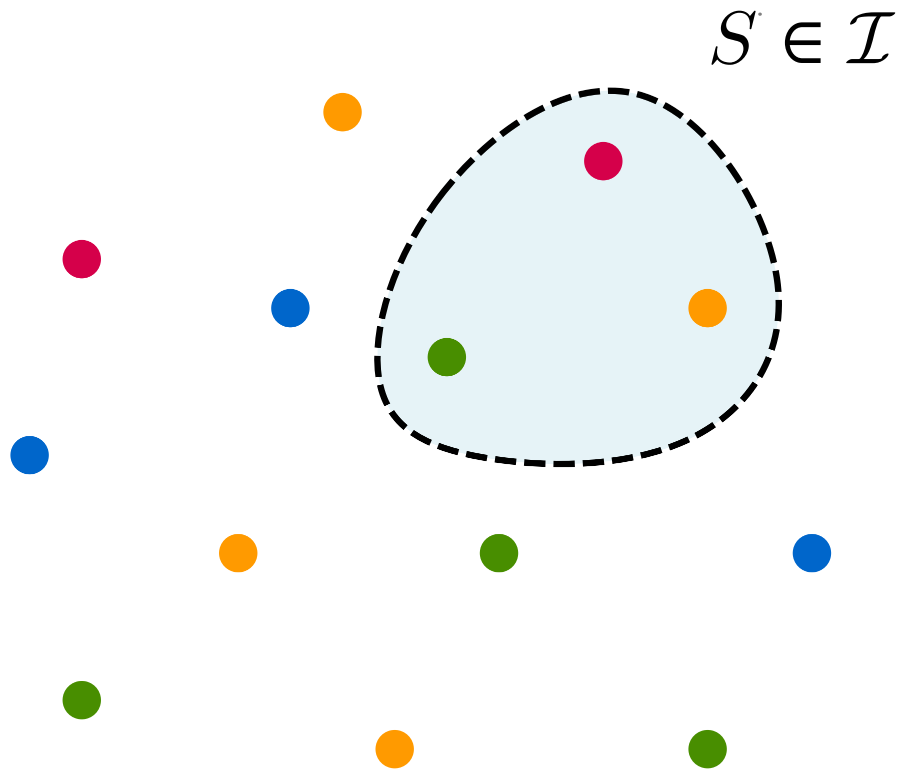

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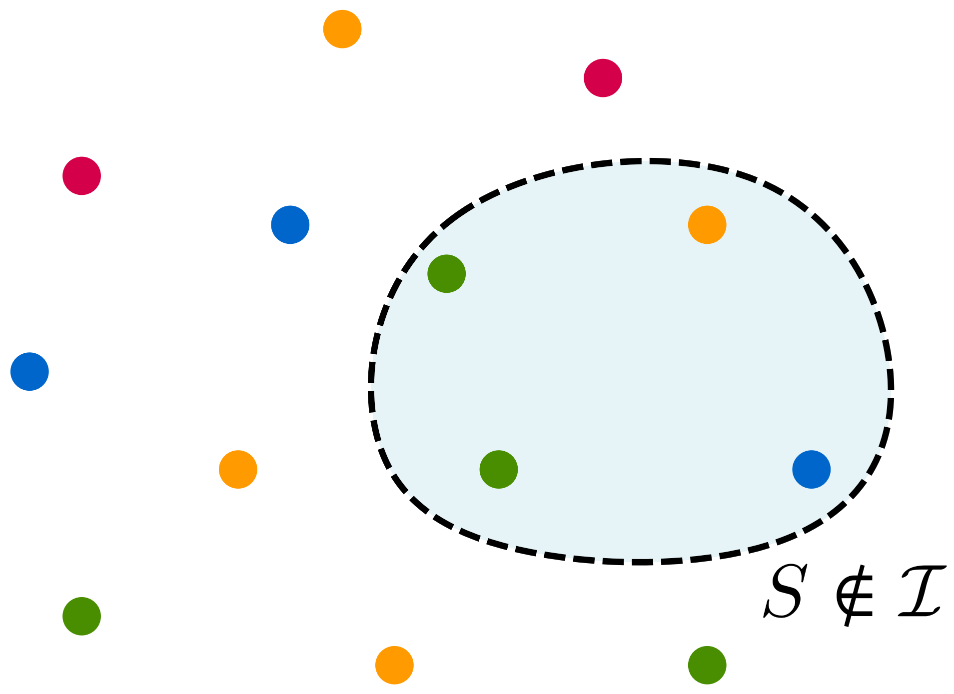

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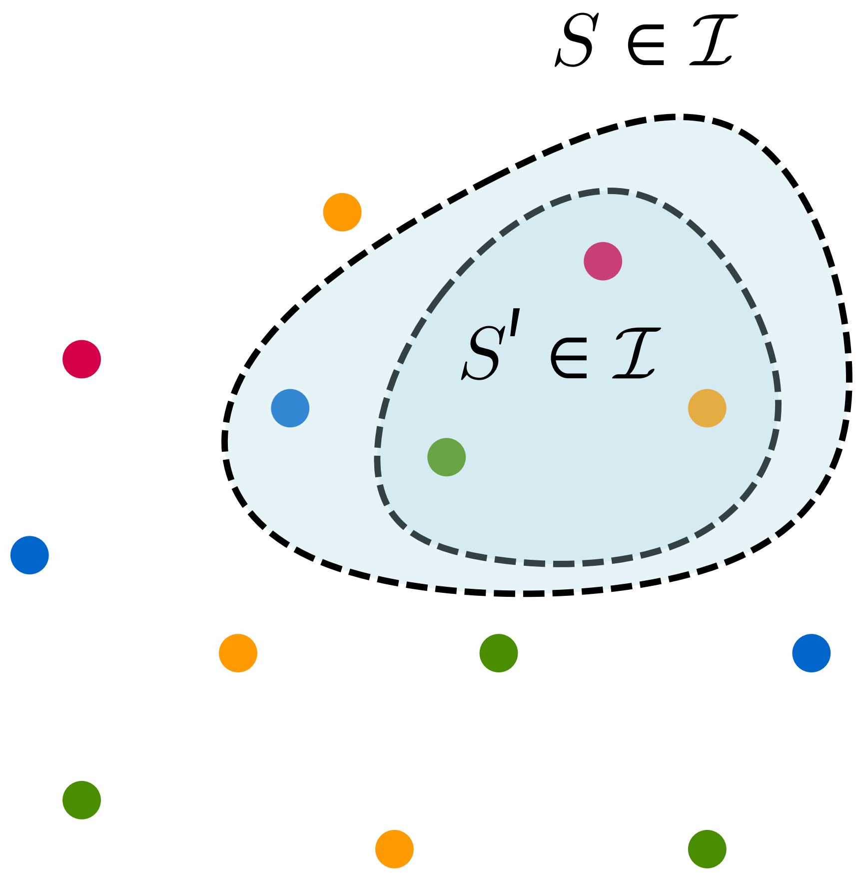

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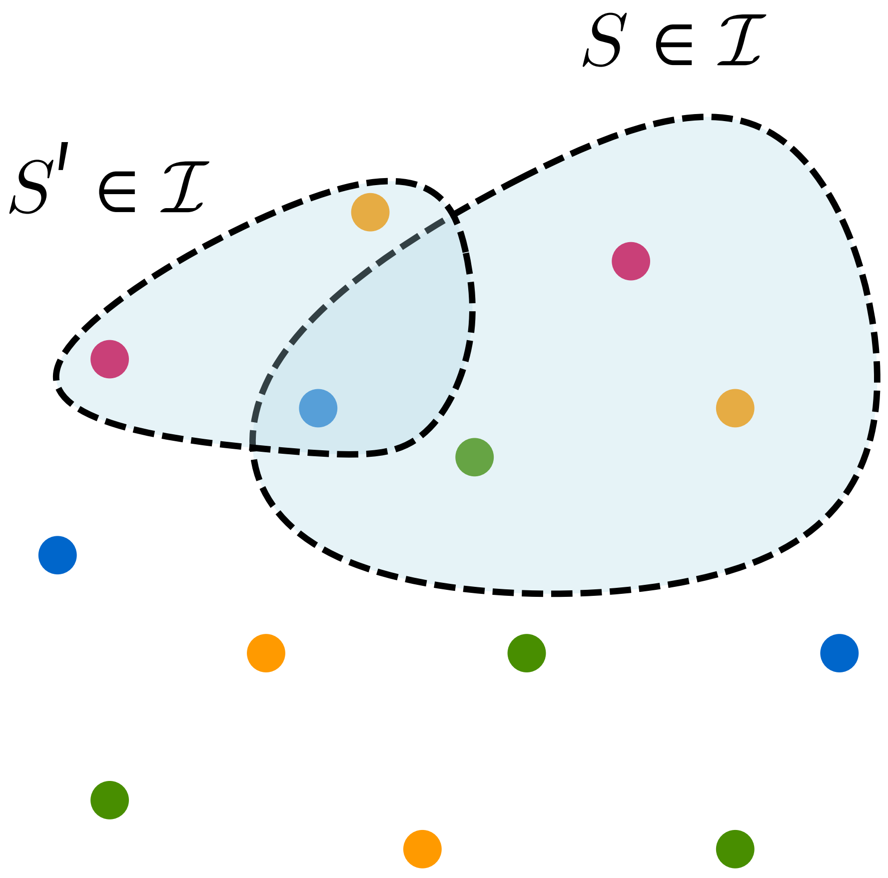
   - Exchange property

   "All maximal independent sets have the same size"



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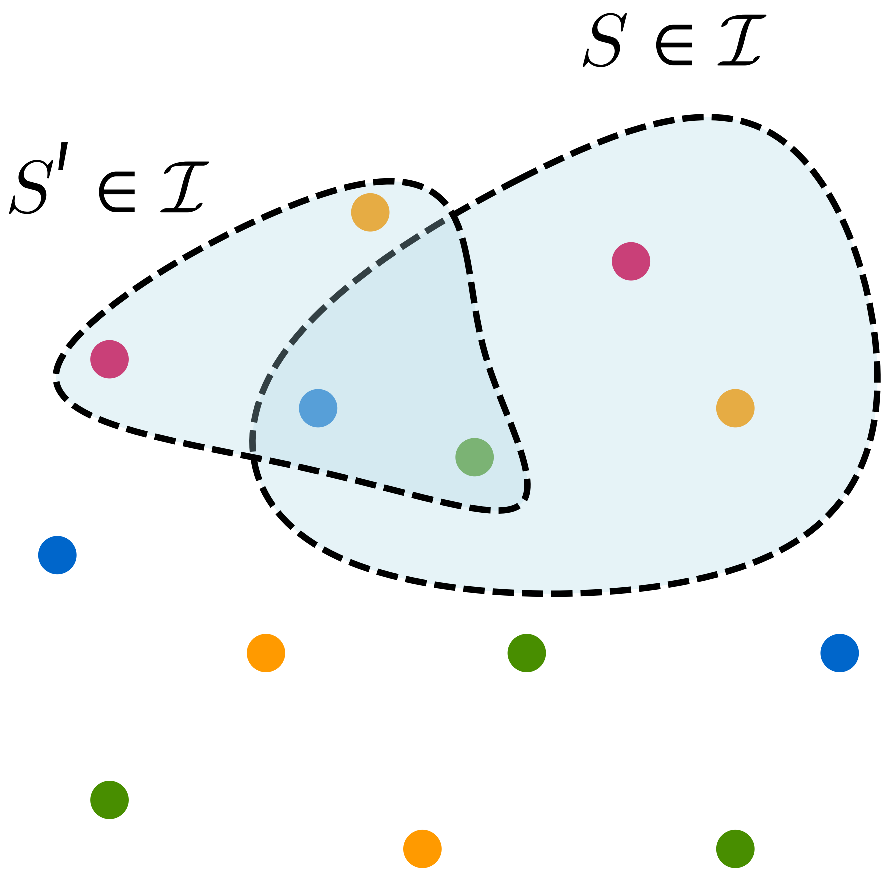

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Matroids: Examples

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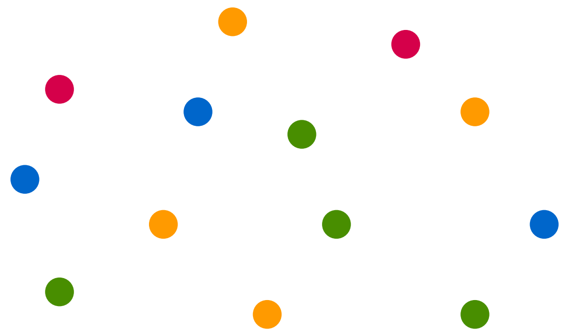

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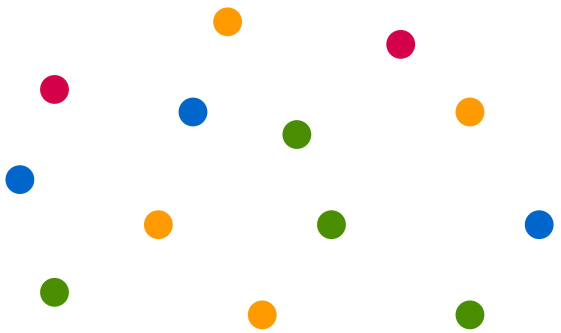

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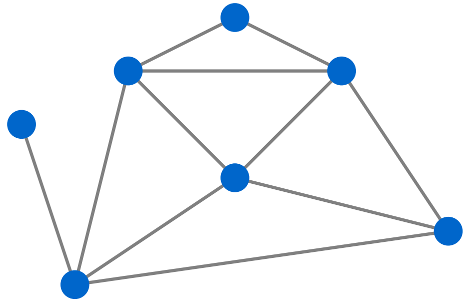
Graphic Matroid



V = edges

$\mathcal{I}$ =“no cycles”

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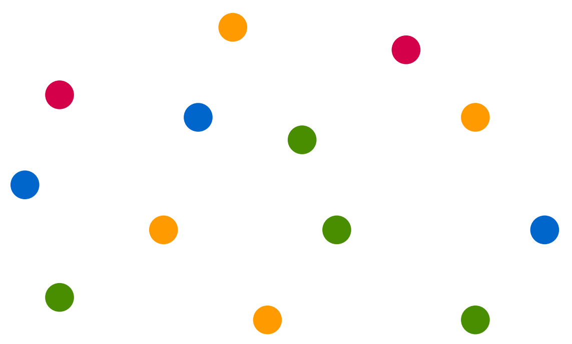

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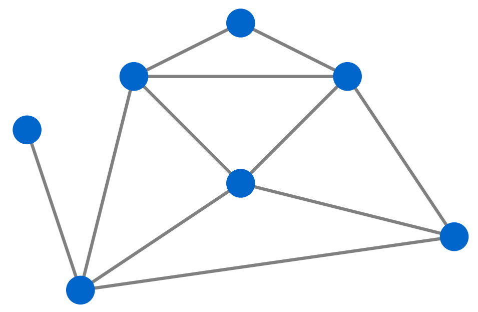

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Linear Matroid

$(2, 1, 4, 2, 3, 3)$

$(1, 0, 1, 0, 1, 0)$

$(3, 1, 5, 2, 4, 3)$

V = vectors

$\mathcal{I}$ ="linear independence"

Matroids: Examples

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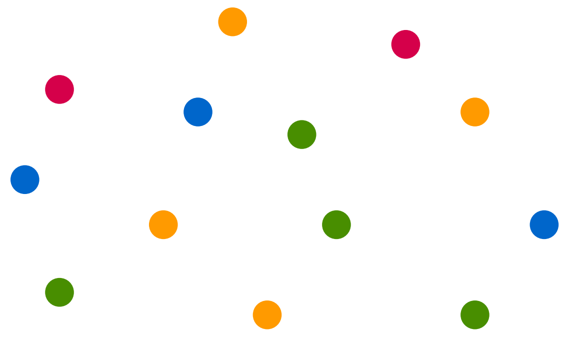

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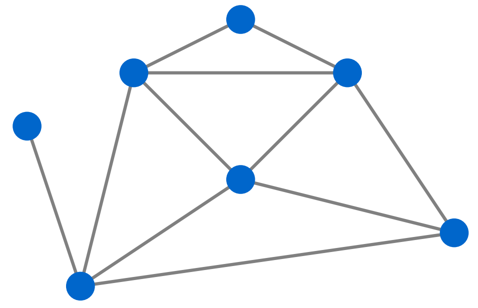

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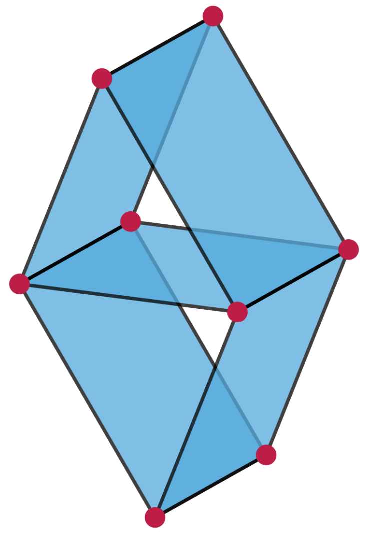

Matroid Intersection

Given two matroids:

- $\mathcal{M}_1 = (V, \mathcal{I}_1)$
- $\mathcal{M}_2 = (V, \mathcal{I}_2)$

Find a *common independent set* $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ of maximum size.

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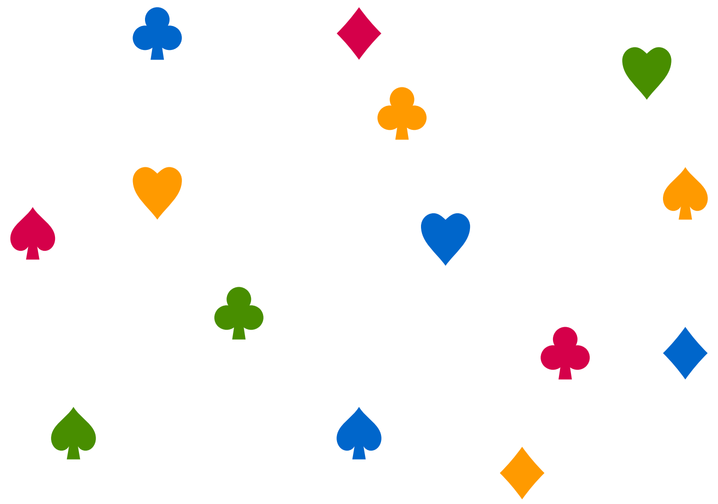

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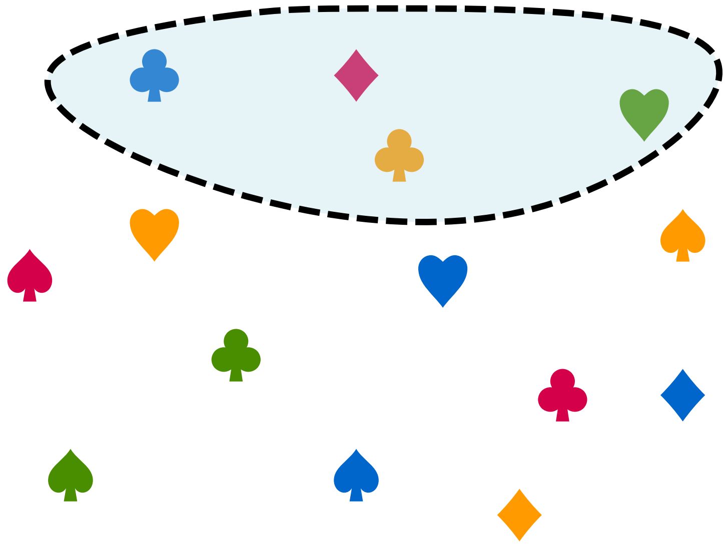

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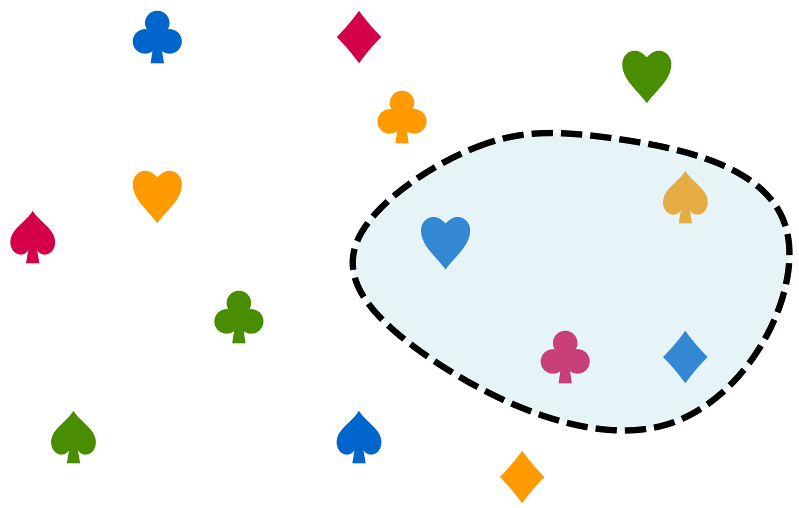

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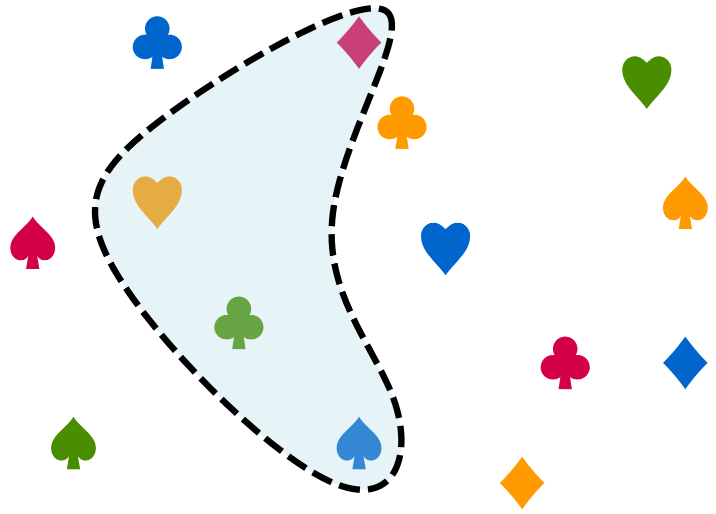

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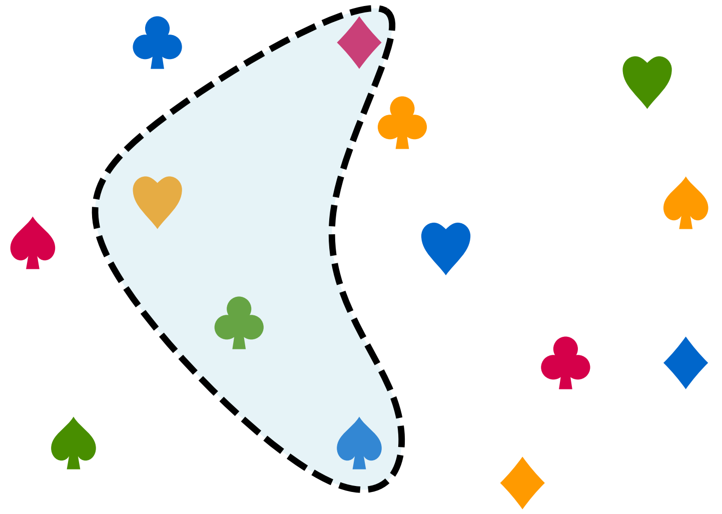

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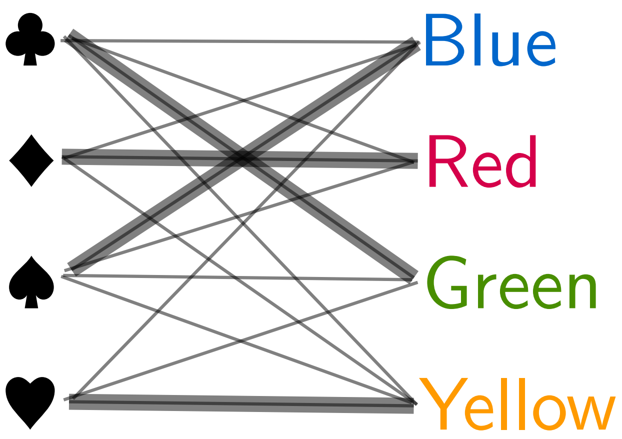


Matroid Intersection: Examples

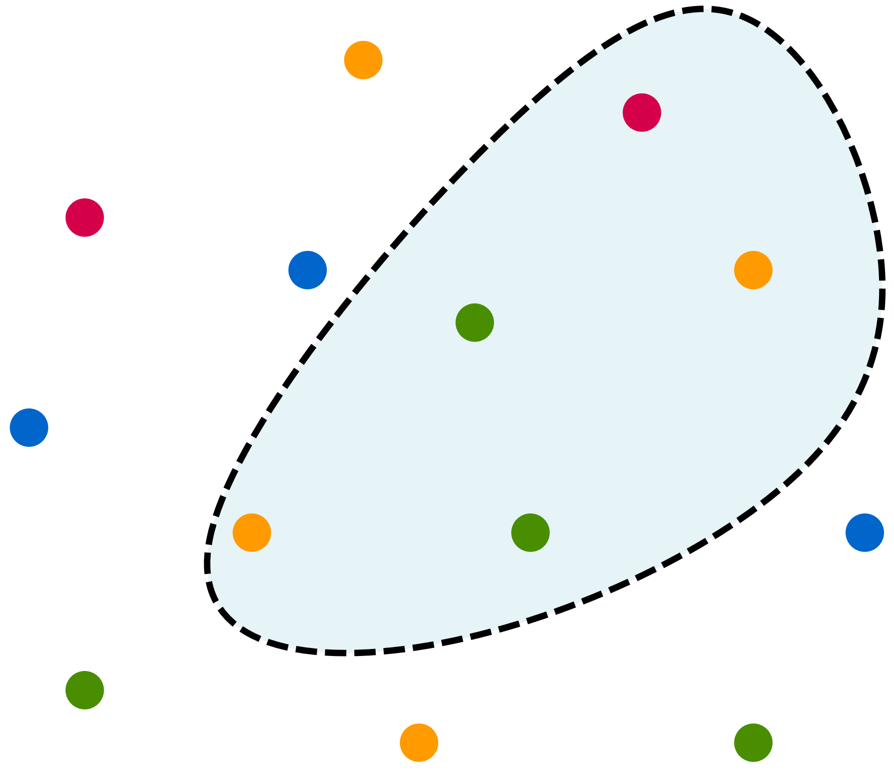
- Bipartite matching
- Arborescence (directed spanning tree)
- Rainbow spanning trees
- Tree/Arborescence packing
- Directed min-cut
- Graph orientation problems
- Matroid partitioning & union
- ...

Also connections to *Submodular Function Minimization*

Matroid Rank

$$\begin{aligned}\mathrm{rk}(S) &= \max\{|A| : A \subseteq S, A \in \mathcal{I}\}\\ &= \text{size of a maximum independent set in } S\\ &= \text{size of a \textit{maximal} independent set in } S\end{aligned}$$

$\mathrm{rk}(S) = 3 =$ #distinct colours



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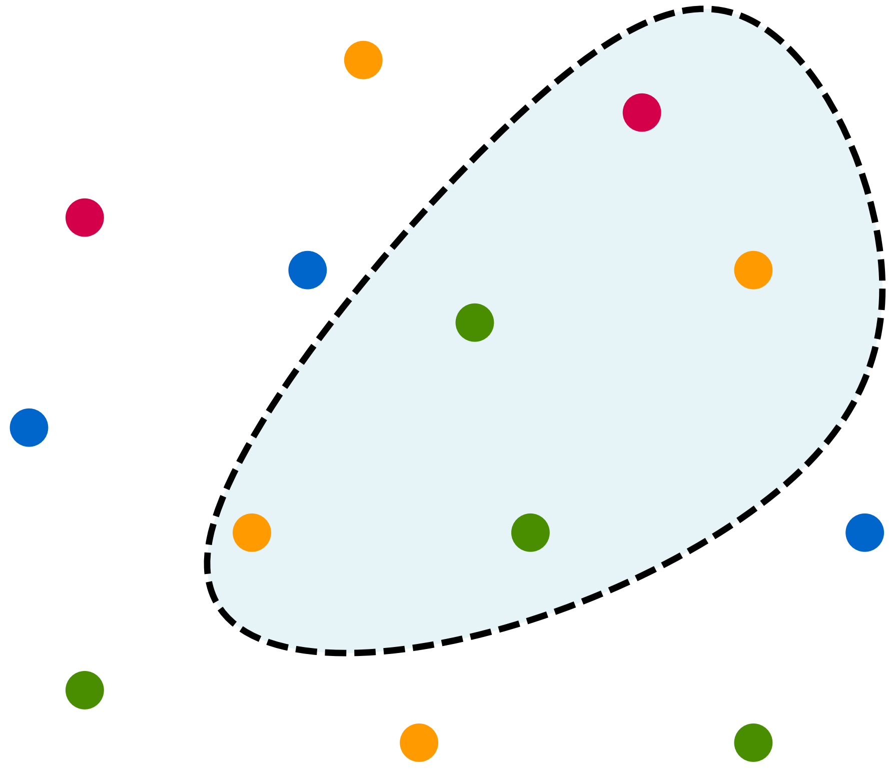


Properties:

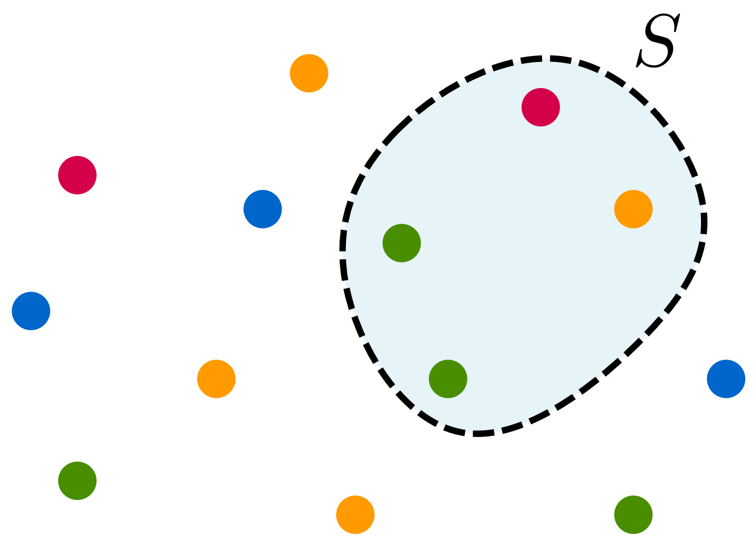
- $S \in \mathcal{I} \iff \mathrm{rk}(S) = |S|$
- Submodular (Diminishing returns)
  If $A \subseteq B$, and $x \notin B$ then:
  $\mathrm{rk}(A+x) - \mathrm{rk}(A) \geq \mathrm{rk}(B+x) - \mathrm{rk}(B)$

Query Access

How to access a matroid?

Oracle Access

- Independence query: "Is $S \in \mathcal{I}$?"
- Rank query: "What is $\mathrm{rk}(S)$?"



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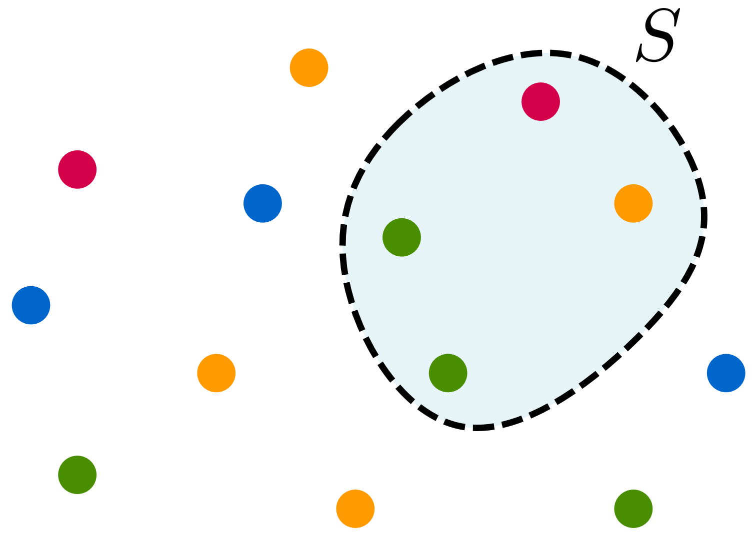

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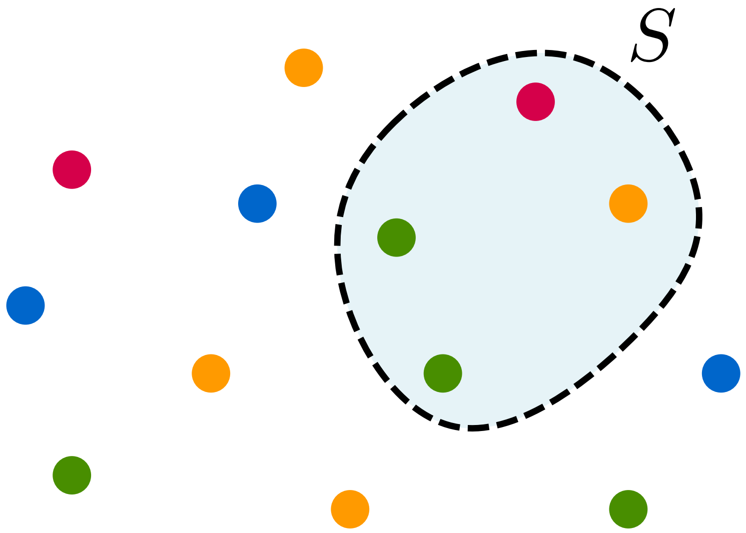

Important:
We do not know the underlying structure of the matroids!

Parallel Query Algorithms

Runs in rounds

Round: Issue a set of k queries simultaneously:

$$\text{“What is } \mathrm{rk}(S_1)?\text{”}, \ \text{“What is } \mathrm{rk}(S_2)?\text{”}, \ldots, \text{“What is } \mathrm{rk}(S_k)?\text{”}$$

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Total number of queries used	$O(n^{3/2})$	$O(\mathrm{poly}(n))$	$O(2^n)$
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Main Question:
How many rounds do we need if we can only use $O(\mathrm{poly}(n))$ queries in total?

Can we do. . .

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	NO	For general matroids: $\tilde{\Omega}(n^{1/3})$ (indep: KUW'86, rank: CCK'21)

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$o(n)$-rounds? YES!

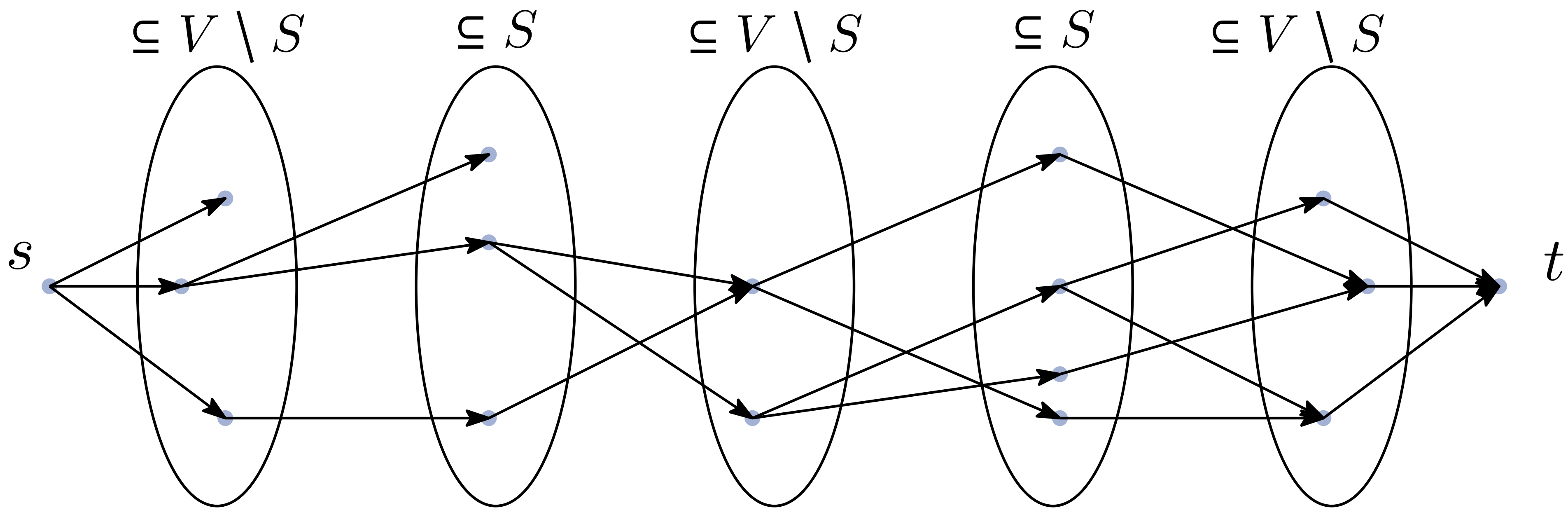
Main Theorem:
Matroid Intersection can be solved using $\mathrm{poly}(n)$ total queries and:

- $O(n^{3/4}) = O(n^{0.75})$ rounds (rank-oracle)
- $O(n^{7/8}) = O(n^{0.875})$ rounds (independence-oracle)

Exchange Graph & Augmenting Paths [Edmonds'60s]

Given $S \in \mathcal{I}_1 \cap \mathcal{I}_2$ construct the *exchange graph* $G(S)$.

s, t-path $\iff$ can increase size of S!



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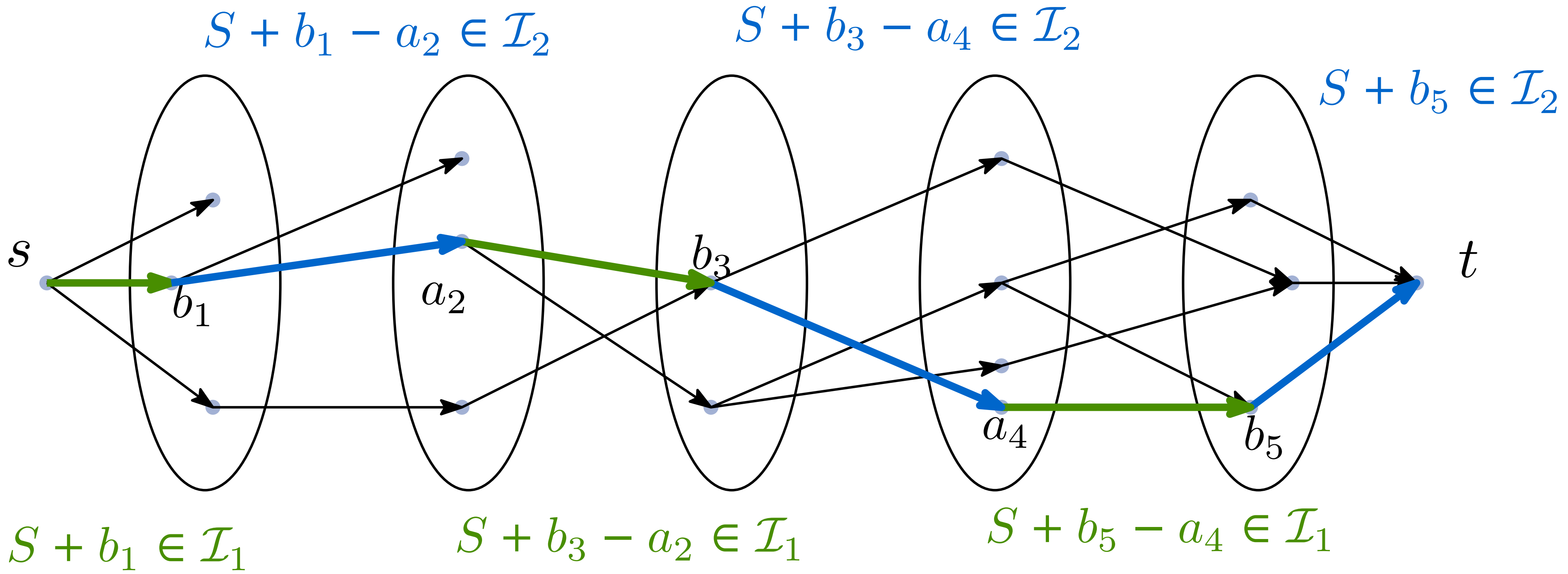

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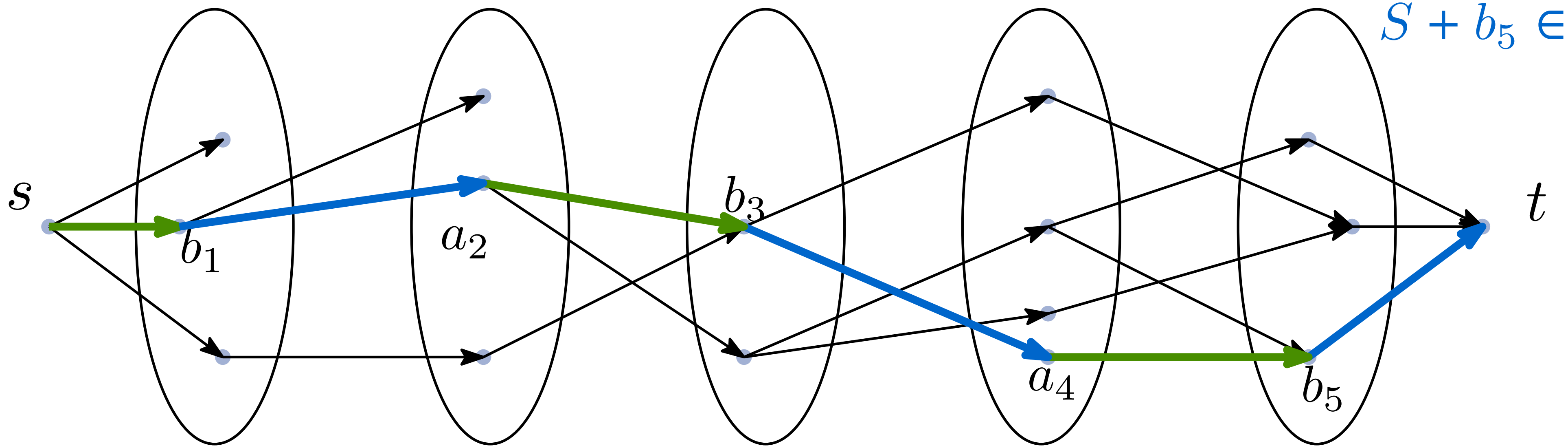
$$\implies S + b_1 - a_2 + b_3 - a_4 + b_5 \in \mathcal{I}_2$$

$S + b_1 - a_2 \in \mathcal{I}_2$ $\quad$ $S + b_3 - a_4 \in \mathcal{I}_2$ $\quad$ $S + b_5 \in \mathcal{I}_2$



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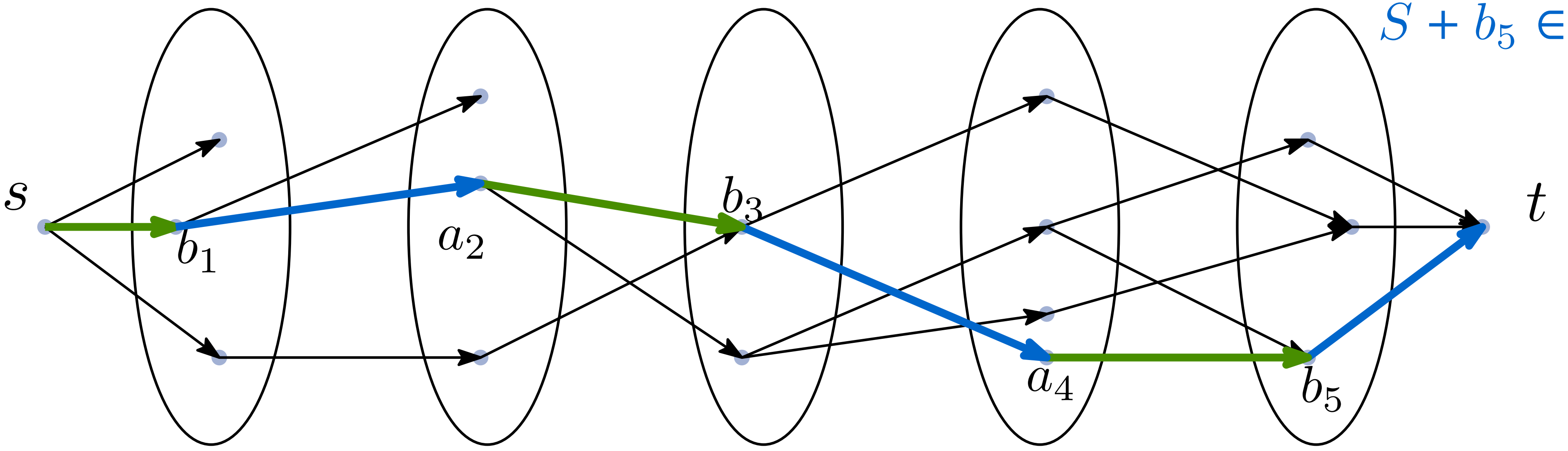

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$$\implies S + b_1 - a_2 + b_3 - a_4 + b_5 \in \mathcal{I}_1$$

Common independent set $S' := S + b_1 - a_2 + b_3 - a_4 + b_5$ of size $|S'| = |S| + 1$

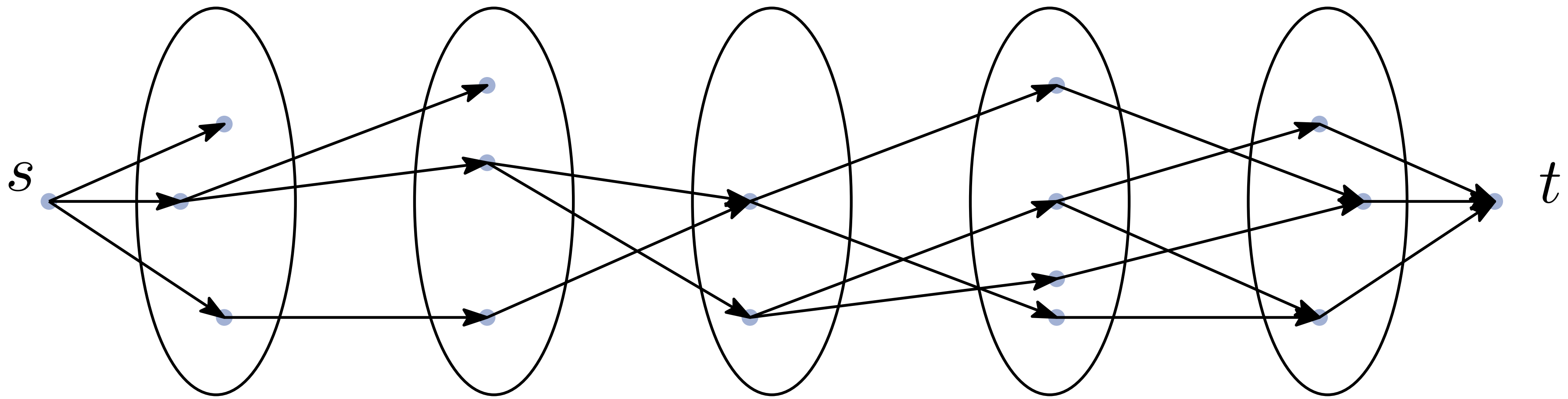
Linear-round Algorithm [Edmonds'60s]

Algorithm $O(n)$ rounds

1. $S = \varnothing$
2. In parallel find all the edges of the exchange graph $G(S)$

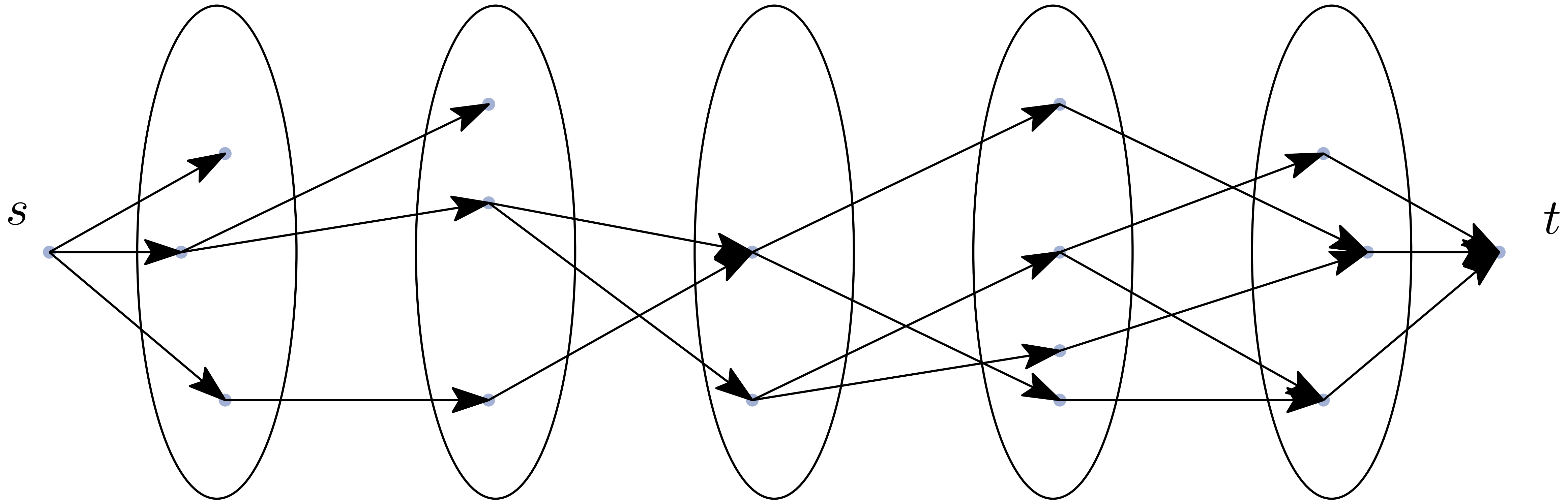
   ▷ 1 round of $O(n^2)$ queries
3. If there is an augmenting path, augment along it and repeat

   ▷ only repeats $O(n)$ times



Augmenting in Parallel?

Exchange graph $G(S)$ behaves weirdly. . .



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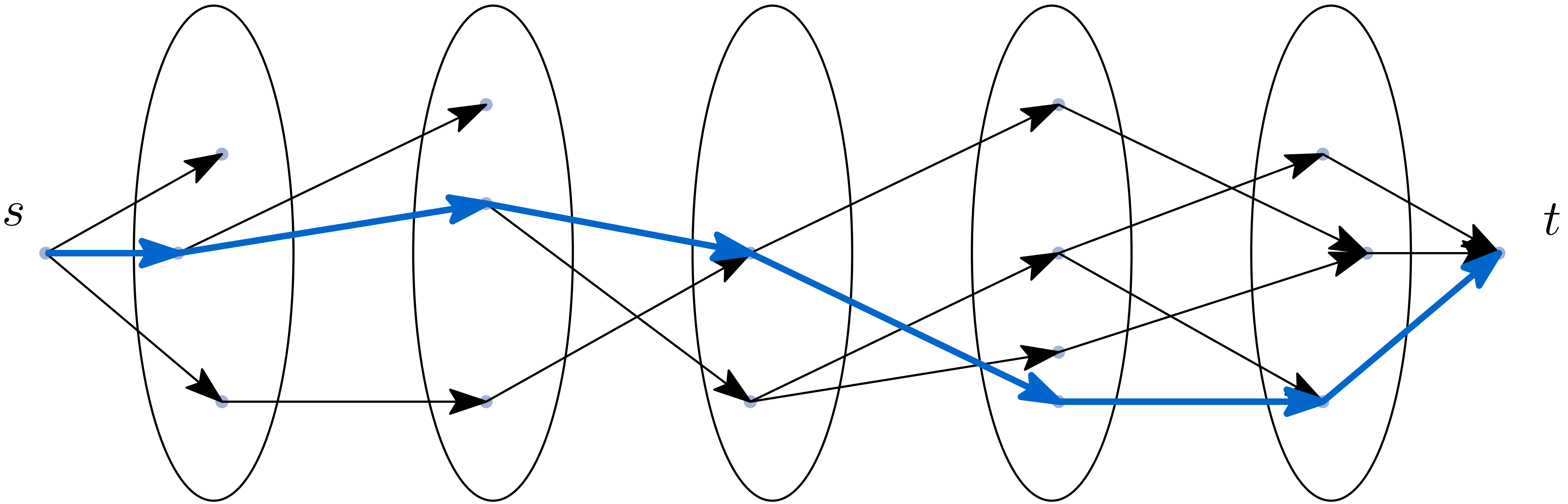

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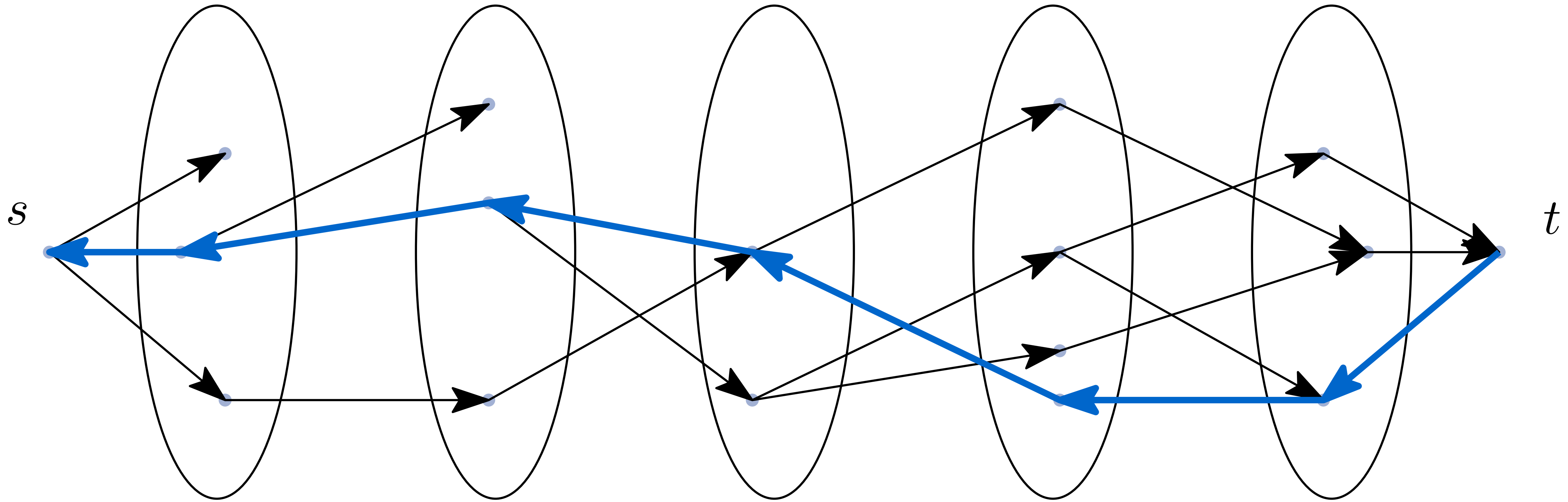

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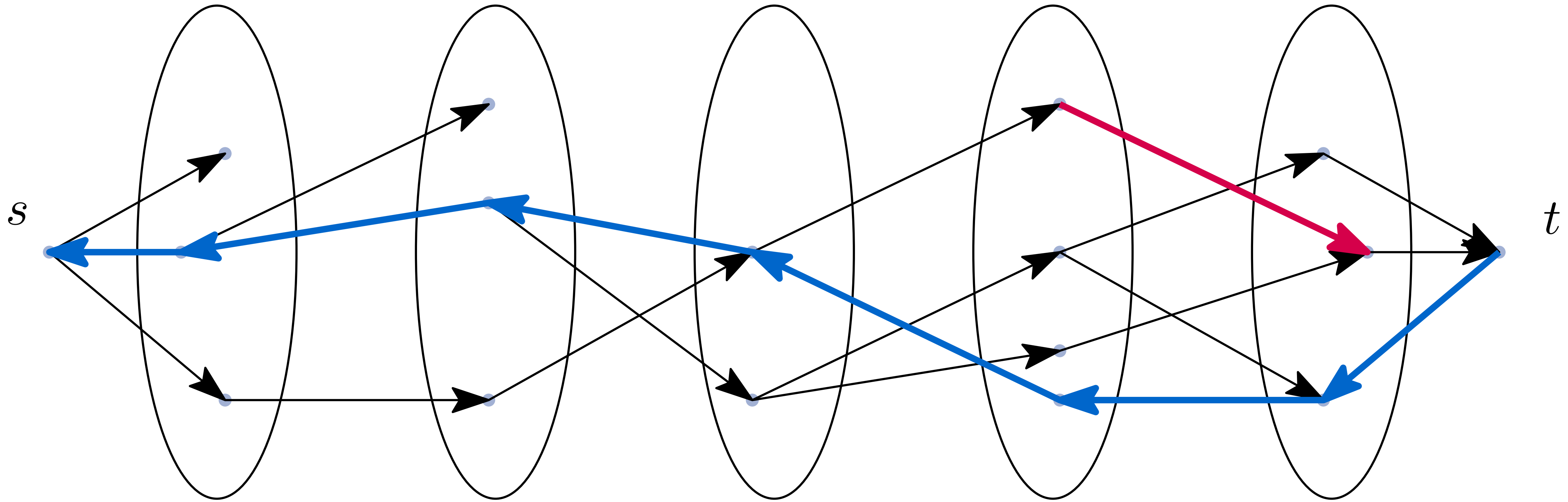

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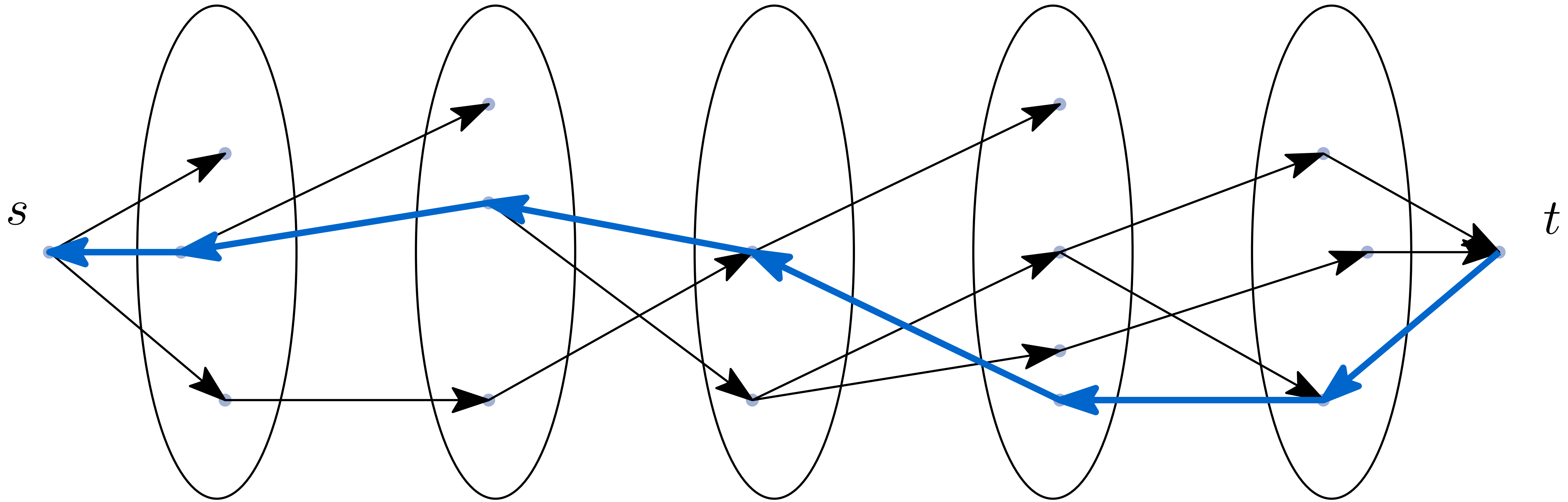

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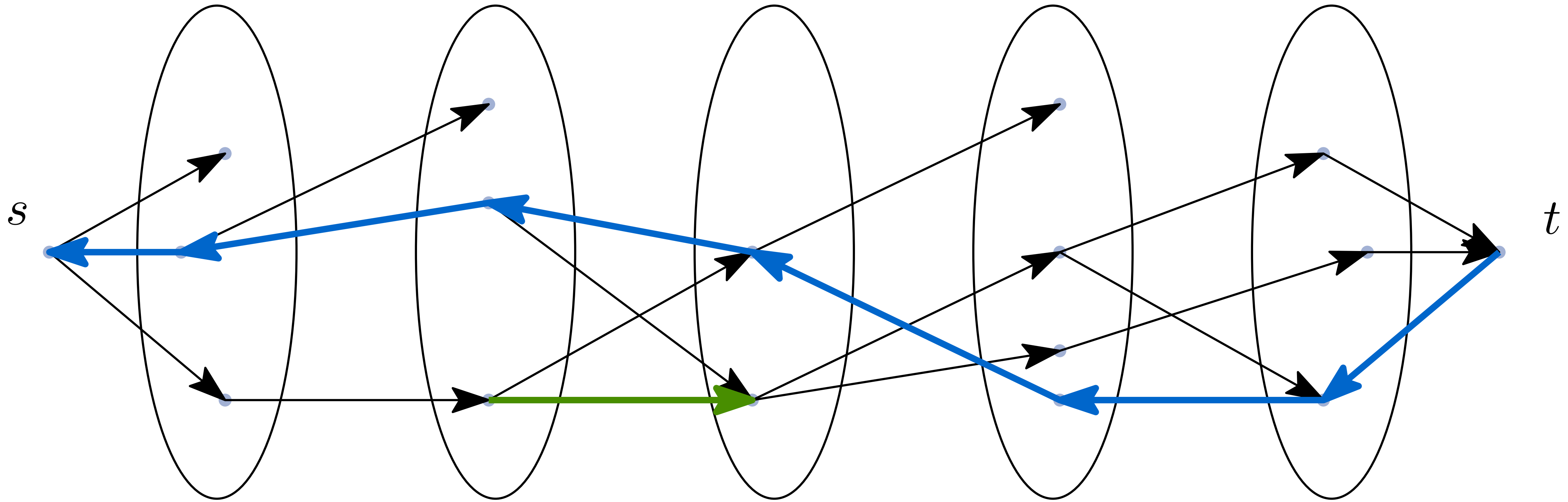
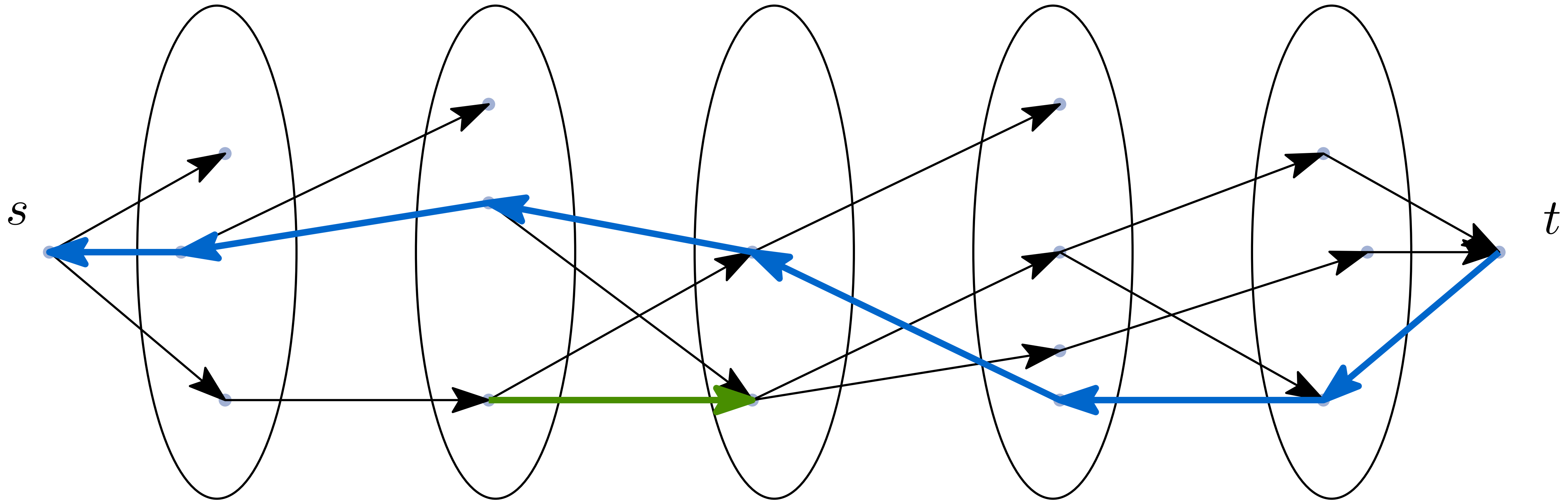


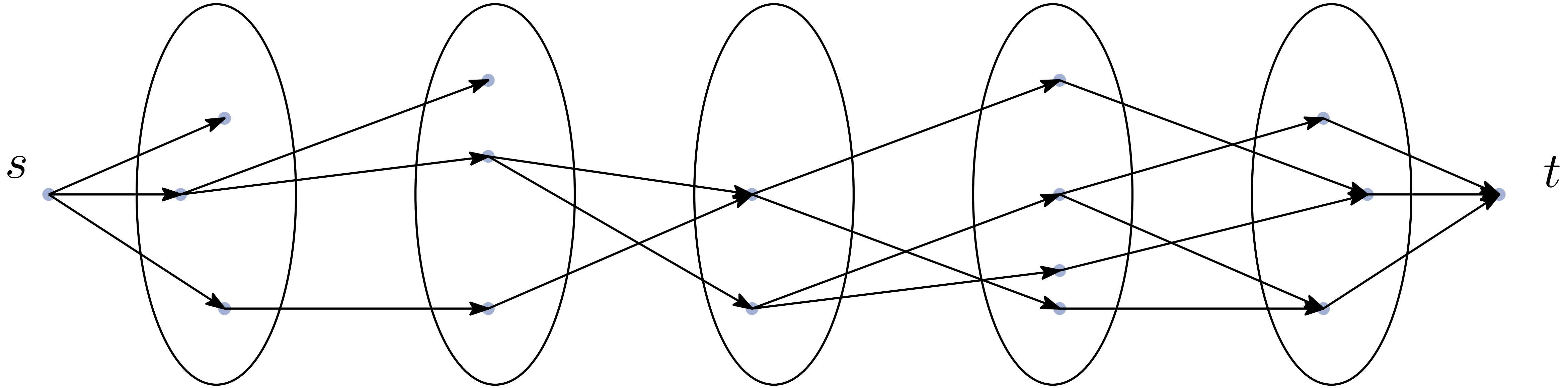
Augmenting in Parallel?

Exchange graph $G(S)$ behaves weirdly...



- Disjoint paths not necessarily "compatible"
- Need to handle the inserted edges

Augmenting Sets [Chakrabarty-Lee-Sidford-Singla-Wong'19]



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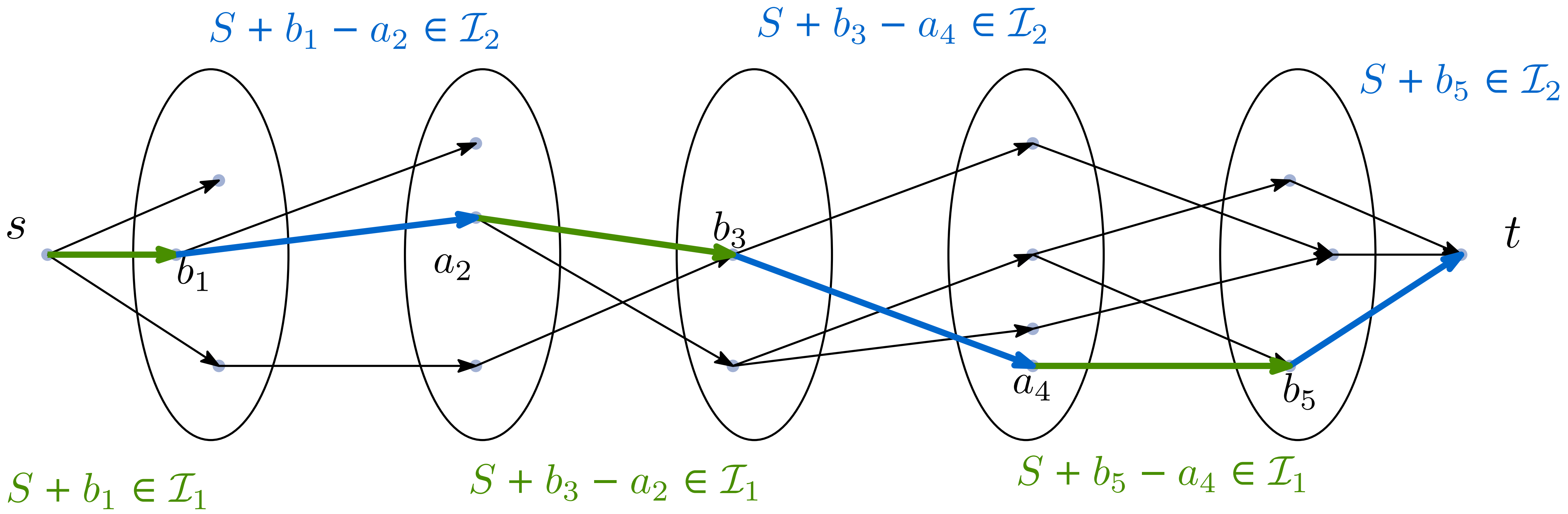

$$\implies S + b_1 - a_2 + b_3 - a_4 + b_5 \in \mathcal{I}_1$$

$$\implies S + b_1 - a_2 + b_3 - a_4 + b_5 \in \mathcal{I}_2$$

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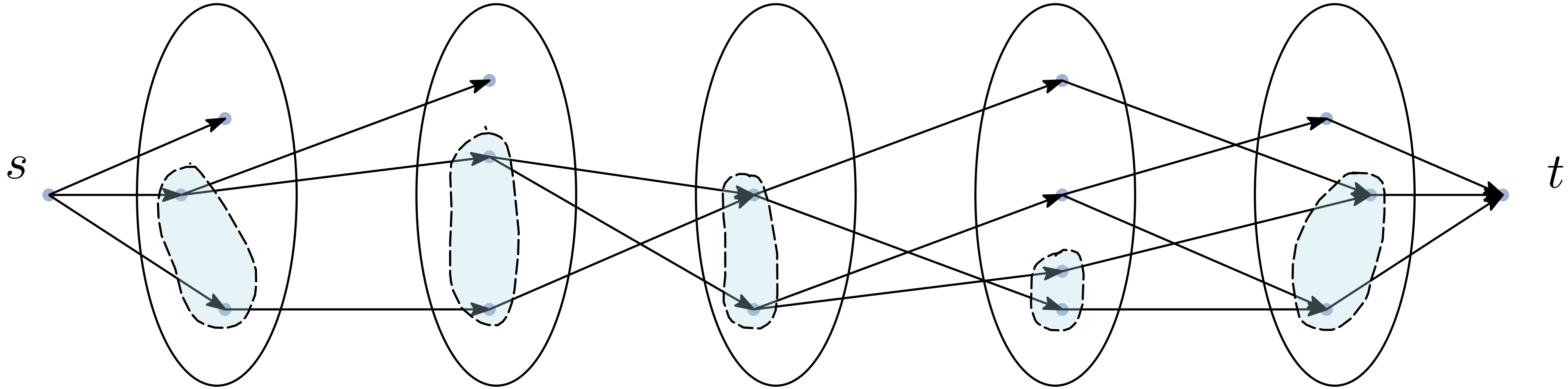
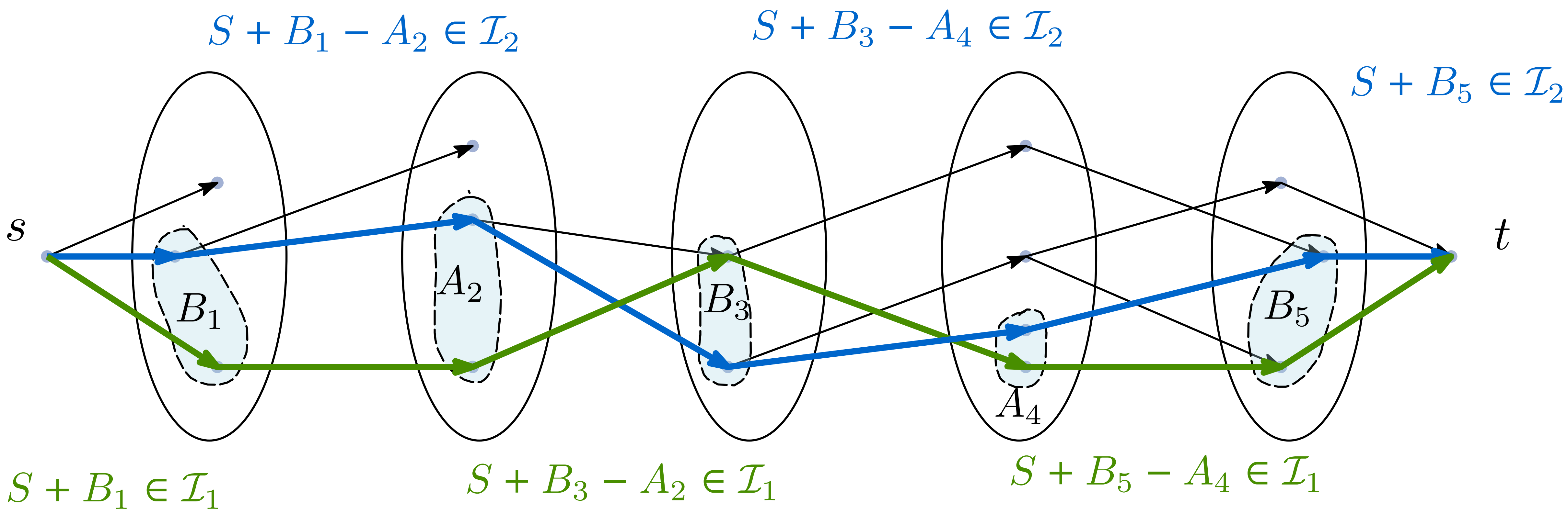


Augmenting Sets [Chakrabarty-Lee-Sidford-Singla-Wong'19]



$$\implies S + B_1 - A_2 + B_3 - A_4 + B_5 \in \mathcal{I}_1 \cap \mathcal{I}_2$$

Sublinear-round Algorithm

Key Lemma ("Blocking-Flow" Approximation Algorithm)
We can get a $(1-\varepsilon)$-approximation of the matroid intersection problem in $O(\sqrt{n}/\varepsilon)$ rounds of $poly(n)$ many rank-queries.

Sublinear-round Algorithm

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Exact Algorithm

1. Run $O(\sqrt{n}/\varepsilon)$-round $(1-\varepsilon)$-approximation algorithm
2. Now we have $|S| \geq \mathrm{OPT} - O(n\varepsilon)$
3. Do these augmentations one-by-one, in a single round each

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1. Run $O(\sqrt{n}/\varepsilon)$-round $(1-\varepsilon)$-approximation algorithm with $\varepsilon = n^{-1/4}$
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$$O(n^{3/4})\text{-round algorithm!}$$

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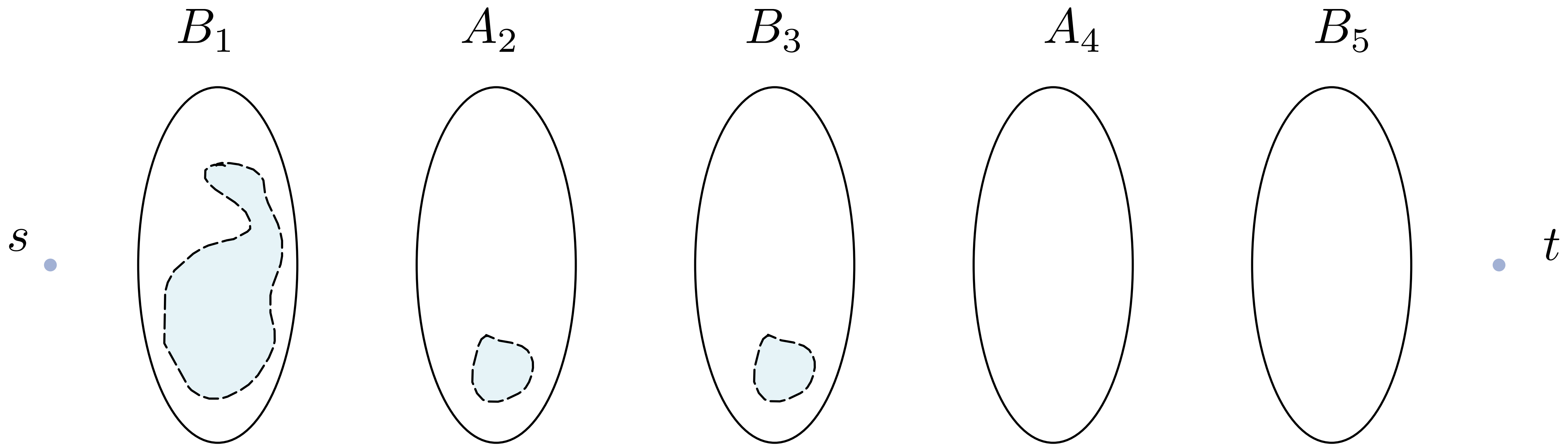
$O(n^{3/4})$-round algorithm!

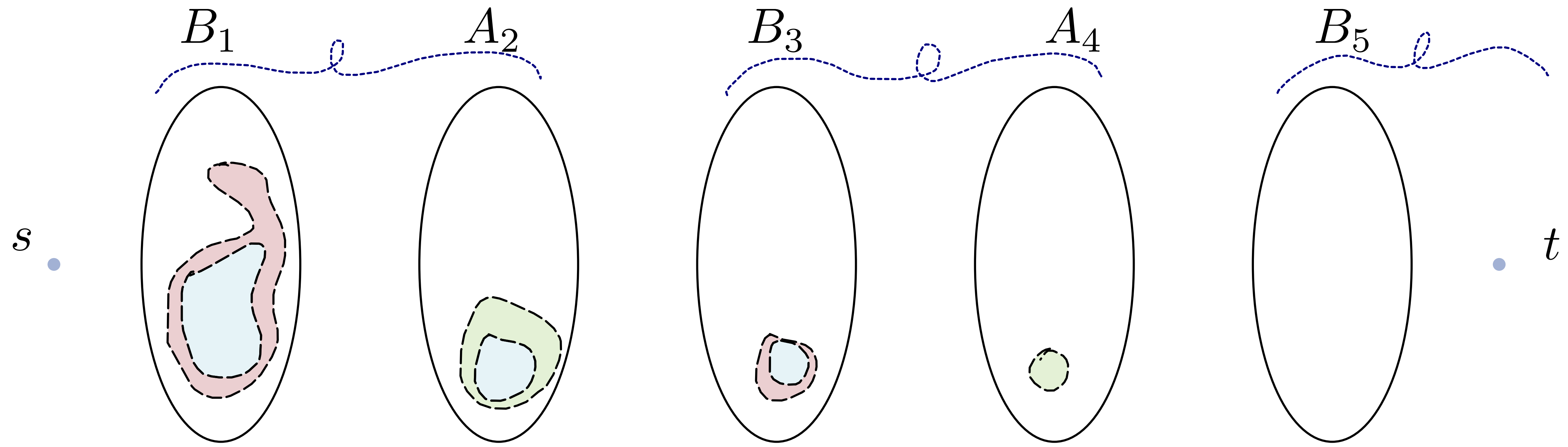
Refining: Combining algorithms of [CLSSW'19] and [KUW'86]

Partial Augmenting Set, "*Staircase*"



Refining: Combining algorithms of [CLSSW'19] and [KUW'86]

In $O(1)$ rounds accessing $\mathcal{M}_2$:



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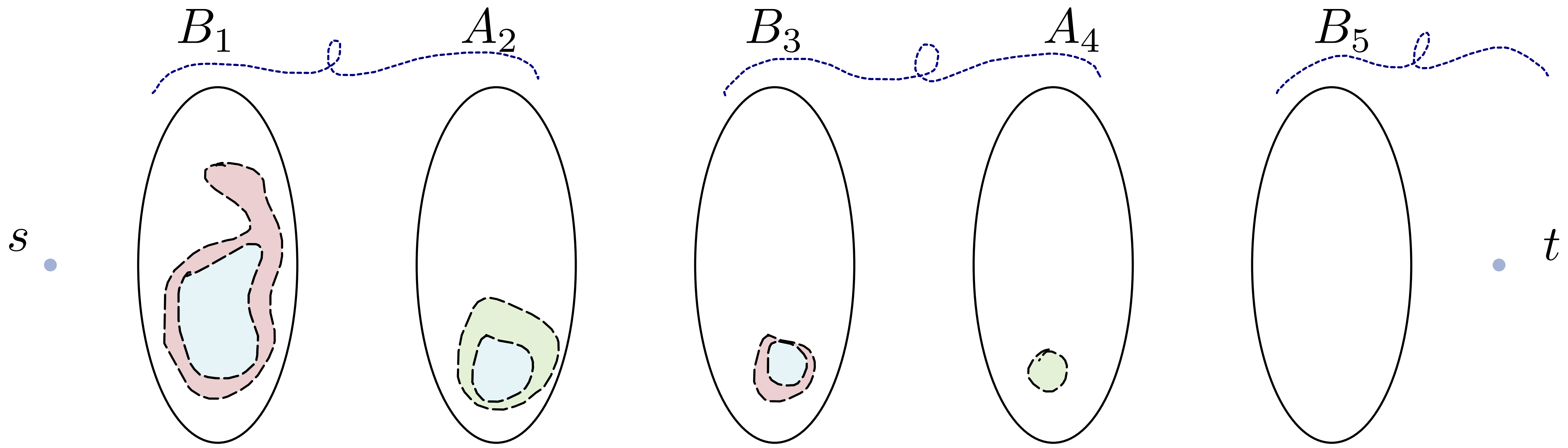

Roughly $|B_1| - |B_\ell|$ elements are newly *discarded* or *selected*

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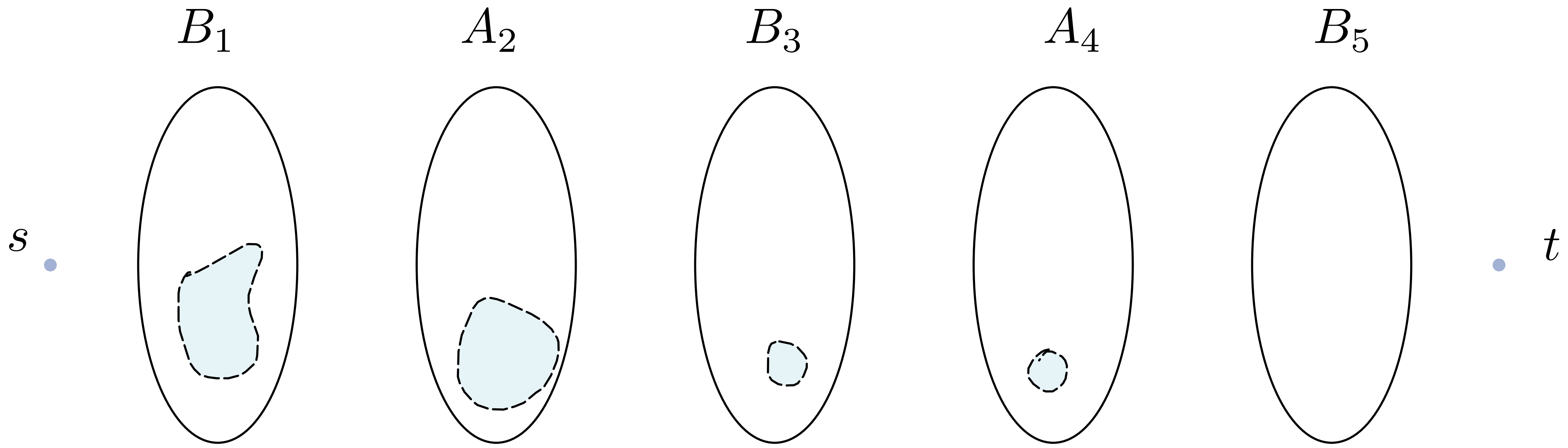

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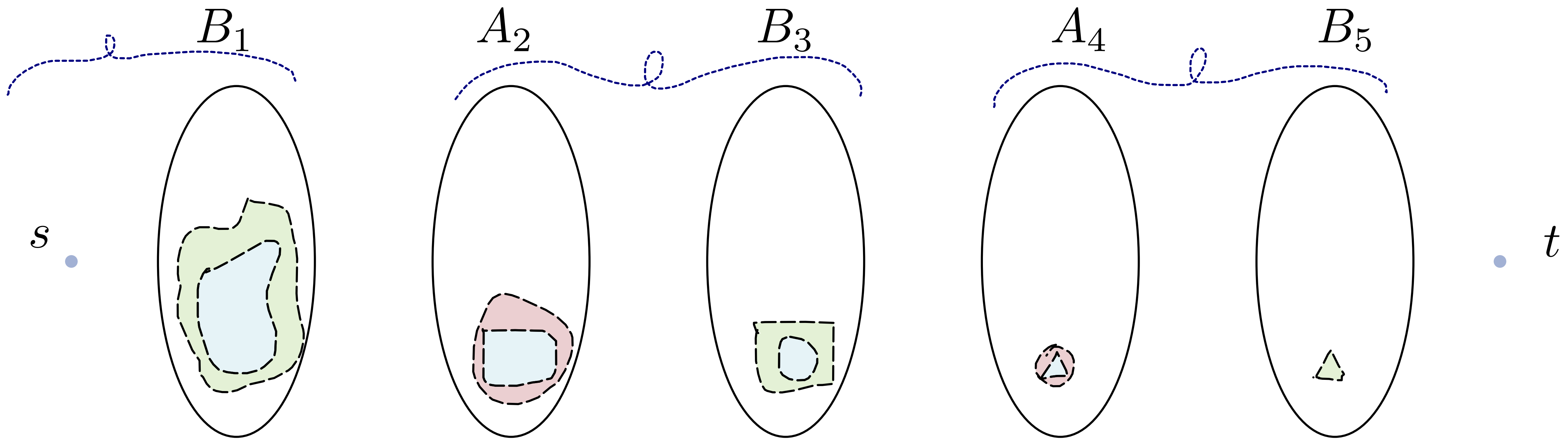
In $O(1)$ rounds accessing $\mathcal{M}_1$:



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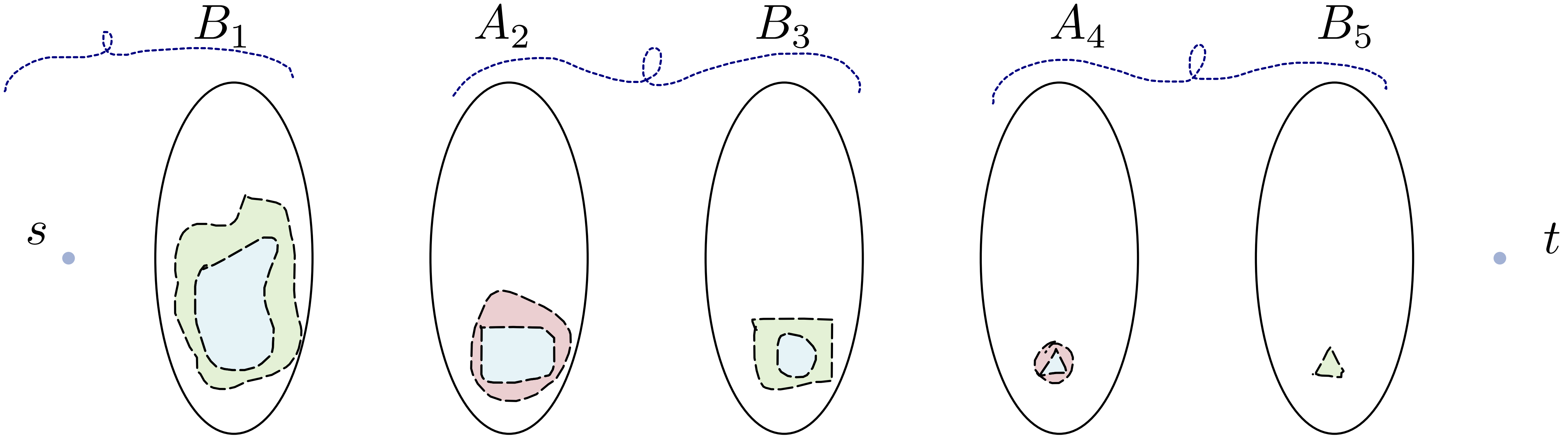

Roughly $|B_1| - |B_\ell|$ elements are newly *discarded* or *selected*

Each element can go *free* → *selected* → *discarded*

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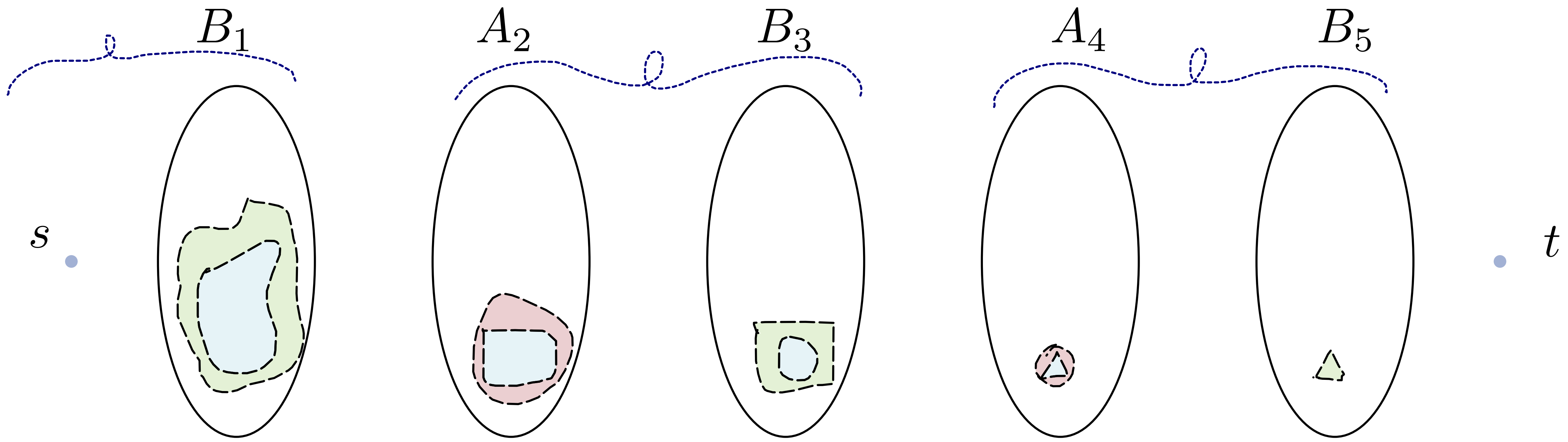
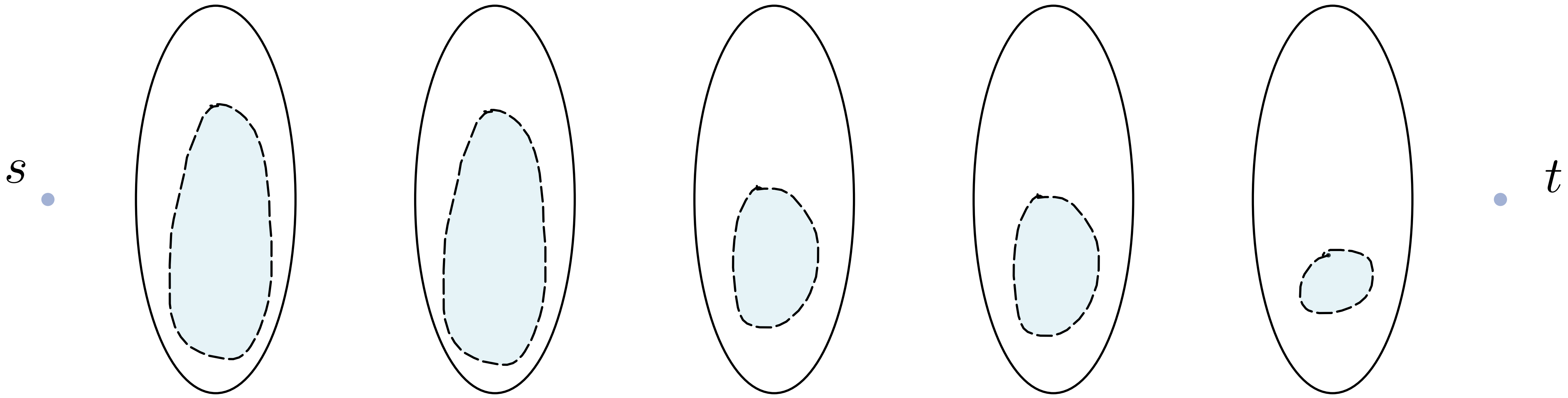


Roughly $|B_1| - |B_\ell|$ elements are newly *discarded* or *selected*

Each element can go *free* → *selected* → *discarded*

Only $\sqrt{n}$ rounds until $|B_1| - |B_\ell| \leq \sqrt{n}$

Falling back to paths



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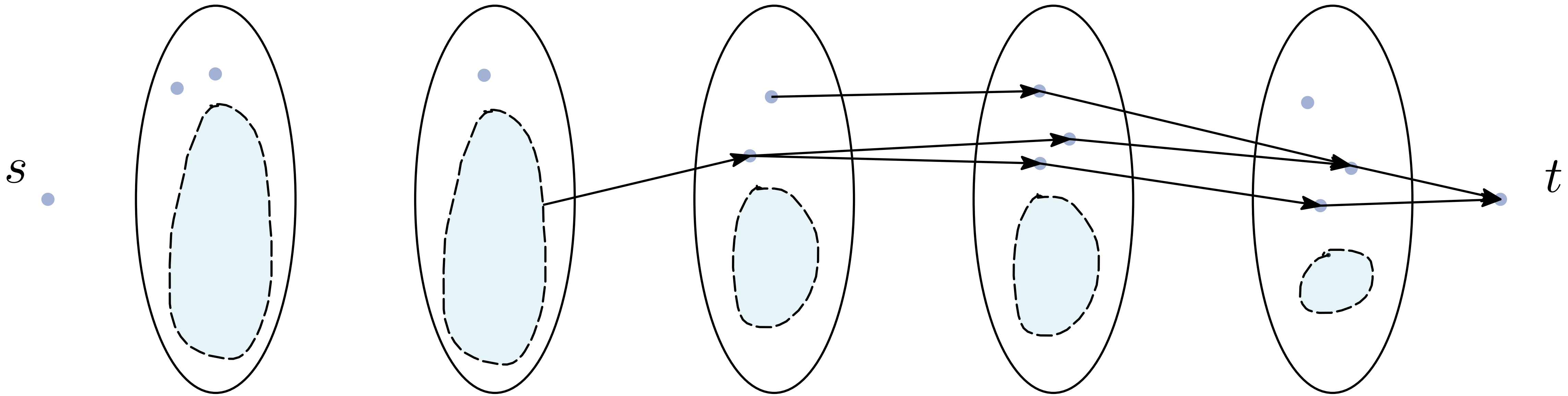

- Can define a graph *with respect to* our “staircase”

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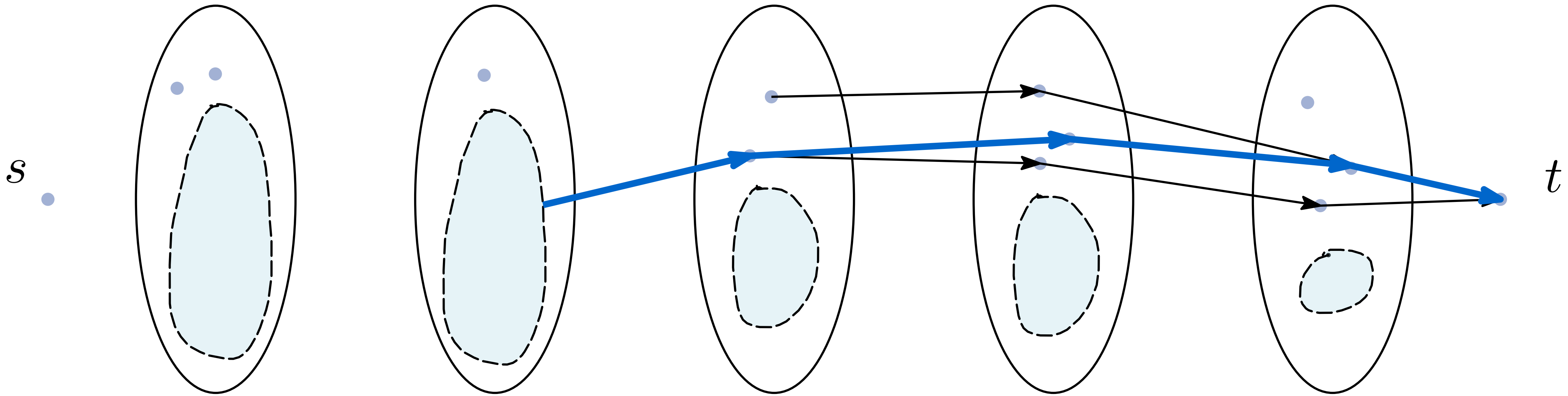

- Can define a graph *with respect to* our “staircase”
- In a single round we can find an “augmenting path”

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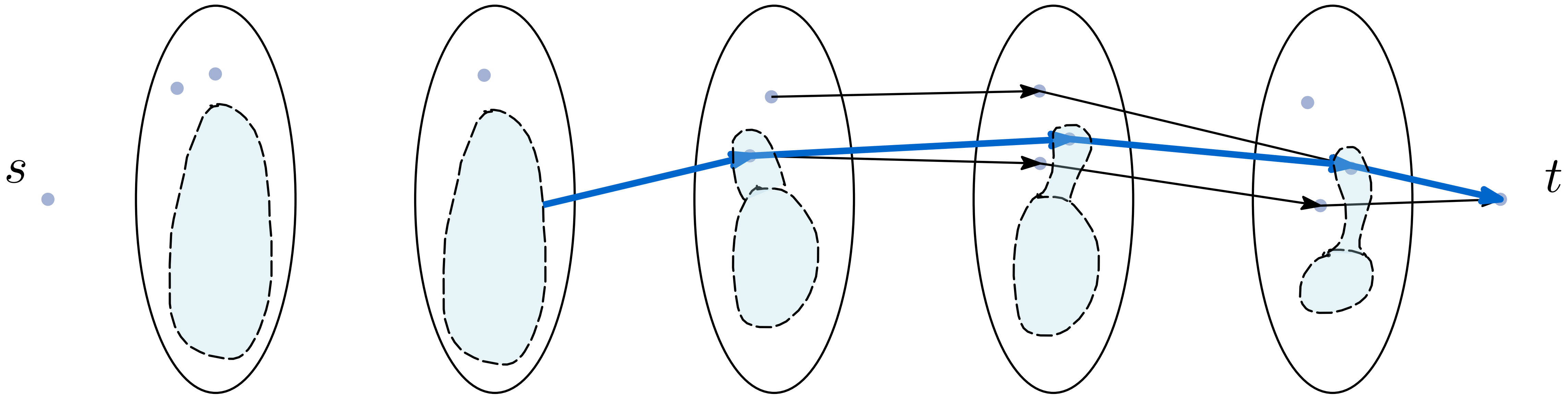

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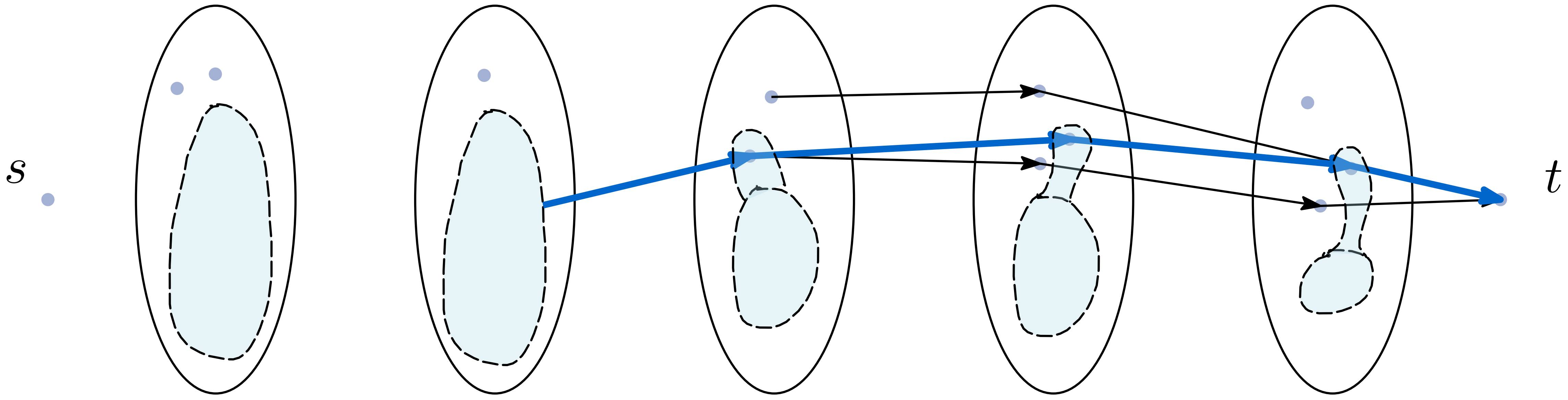

- Can define a graph *with respect to* our “staircase”
- In a single round we can find an “augmenting path”
- Only need to repeat $O(\sqrt{n})$ times

Open Problems

- What is the actual number of rounds required?
  - Somewhere between $\tilde{\Omega}\left(n^{1/3}\right)$ and $O\left(n^{3/4}\right)$. $\sqrt{n}$?
- What about *submodular function minimization* (SFM)?
- What about *weighted* matroid intersection?
  - Similar $O(n)$ one-by-one algorithm works.
  - Can we also acheive sublinear number of rounds?

Thanks!